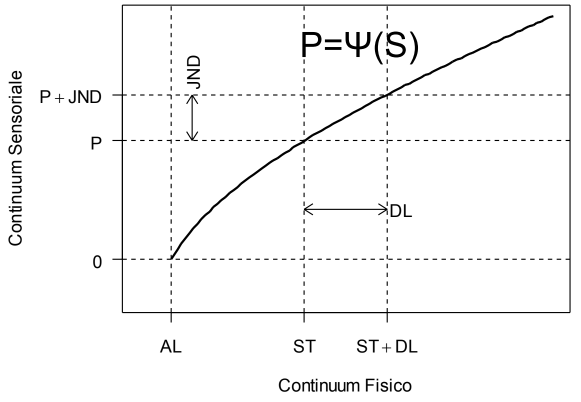
Sensory thresholds

Gabriel Baud-Bovy

Fondamental definitions



- The **psychophysical function Ψ** relates the physical intensity of the stimulus to the corresponding sensation. Note that the sensations are not directly observable.

- The **absolute threshold** (*Absolute Limen, AL*) is the smallest amount of stimulus energy necessary to produce a sensation
- The **difference threshold** (*Difference Limen, DL*) is the amount of changes of a stimulus required to produce a **just noticeable difference** (JND) in the sensation. Note that difference thresholds refer to the stimuli while JND refer to sensations.

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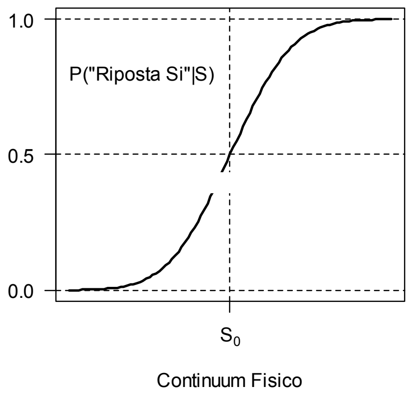
The psychometric function

- In experiments aimed at measuring thresholds, the stimulus intensity is systematically manipulated.



- The psychometric function is the probability of some observer's **response** as a function of the **stimulus** intensity.

$$F(S) = \text{Prob}(R \mid S)$$

- Typically, psychometric functions have an ogival (sigmoidal) shape.

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Response matrix

Conjoint probabilities

	R_1	...	R_N	
S_1	$P(R_1 \cap S_1)$		$P(R_N \cap S_1)$	$P(S_1)$
...				
S_M	$P(R_1 \cap S_M)$		$P(R_N \cap S_M)$	$P(S_M)$
	$P(R_1)$	...	$P(R_N)$	

Marginal probabilities

$$P(S_i) = \sum_{j=1}^{N} P(R_j \cap S_i)$$

$$P(R_j) = \sum_{i=1}^{M} P(R_j \cap S_i)$$

Conditional probabilities

	R_1	...	R_N
S_1	$P(R_1 \mid S_1)$		$P(R_N \mid S_1)$
...			
S_M	$P(R_1 \mid S_M)$		$P(R_N \mid S_M)$

- Many psychophysical experiments involve the repeated presentation of a limited number of of stimuli $\{S_1,..,S_M\}$ and a limited set of possible responses $\{R_1,...,R_N\}$.
- The **response matrix** is the matrix conditional probabilities Prob($R_j|S_i$) where Prob($R_j|S_i$) is the probability of the response Rj for the stimulus Si.
- The outcome of the experiment can be described by the conjoint probability P(R_j and S_i).

$$\text{Prob}(R_j \text{ and } S_i) = \Pr(R_j \mid S_i)P(S_i)$$

- Unlike the matrix of conjoint probabilities, the response matrix is a description of the observer, not of the experiment. As a matter of fact, the conjoint probabilities depends not only on the observer but also on the the probability of the stimulus P(S), which is under control of the experimenter.

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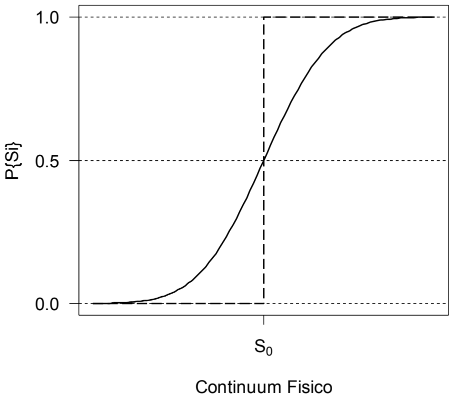
Psychophysical tasks

Psychophysical tasks

- Many psychophysical tasks that differ in subtle ways exist. Methods of analysis (e.g., definition of sensory tresholds) are in general task specific.
- **Detection tasks** are used to measure the absolute thresholds. The observer's task is to indicate whether he has detected the stimulus.
  - Yes-no procedure
  - 2AFC detection task
- **Discrimination tasks** are used to measure difference thresholds. In these tasks, the experimenter present two stimuli, a reference (*standard*) stimulus and a *comparison* stimulus. The task of the subject is to indicate whether he has perceived a difference between the two stimuli.
  - 2AFC discrimination task
  - discrimination task with reminder
  - same-different task
- **Identification o classification tasks** where the observer must identify the stimulus.
- The same taks can be used to build **psychophysical scales,** but specific methods for that purpose also exist (see scaling methods).

Yes-no detection task

- In a simple **Yes-No detection task**, the experimenter presents a stimulus during a an interval and the subject must indicate wether is has detected the stimulus or not.



- The **absolute threshold** (or Absolute Limen, AL) is defined as the value of the stimulus that elicits 50% of positive responses:

$$\text{Prob}(R = \text{“Yes"} \mid S = AL) = 0.5$$

- The outcome of the Yes-No detection task is strongly influcend by the response bias of the observer. For this reason, this task is little used nowdays.

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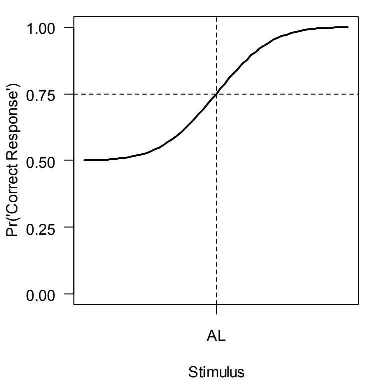
2AFC detection task

- **Two alternative forced choice task**: The experimenter presents the stimulus during one of two intervals (time-separated presentations) or at one of two possible locations (space-separated presentations) and the subject must indicate in which interval or location the stimulus is present.

  The position of the stimulus in space or in time must be randomized between trials.



- The **psychometric function** relates the proportion of **correct responses** as a function of the stimulus intensity. The psychometric function ranges from 50% to 100% because observers respond randomly when the intenisty of the stimulus is below threshold.
- The **absolute threshold** is defined as the stimulus intensity that elicit a 75% of correct response (a different proportion can be used as long as it is larger than 0.5)

- Unlike the yes-no task, the 2AFC task is little influenced by the response bias because the response is based on the *comparison* of two stimuli. Any bias present in the evaluation of the magnitude of a stimulus will affect both stimuli equally and is thus cancelled when the two stimuli are compared.

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Absolute thresholds

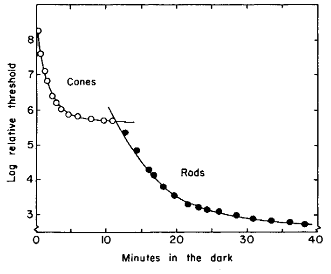

FIG. 2.1. Biphasic curve for dark adaptation. The logarithm of the threshold intensity is plotted against time in the dark. (From Hecht, Haig, & Chase, 1937.)

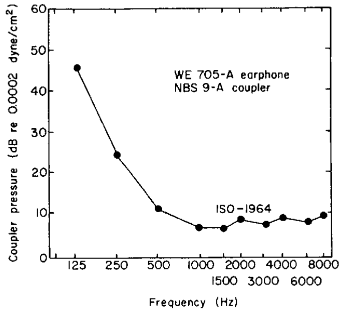

FIG. 2.3. Absolute threshold in decibels sound pressure level for the detection of pure tones as a function of stimulus frequency. (From Davis & Krantz, 1964.)

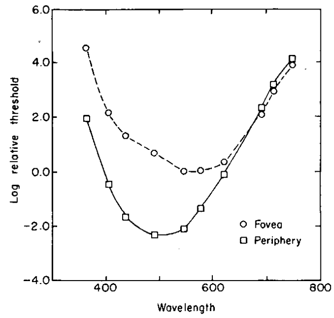

FIG. 2.2. Relative thresholds for detection of light as a function of wavelength and location of the stimulus on the retina. (From Wald, 1945. Copyright 1945 by the American Association for the Advancement of Science.)

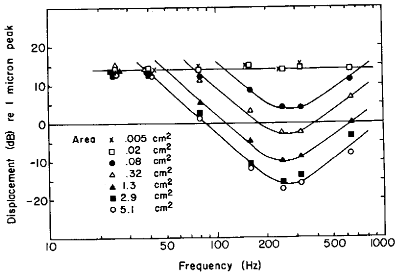

FIG. 2.4. Vibrotactile thresholds for seven contactor sizes as a function of vibration frequency. (From Verrillo, 1963.)

All these examples show large variations of the absolute thresholds

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Discrimination tasks

- **Discrimination tasks** are used to measure difference thresholds. In these tasks, the experimenter present two stimuli, a reference (*standard*) stimulus and a *comparison* stimulus, separated in time or space. The order or position of the stwo stimuli must be randomized to counterbalance possible systematic effects due the their temporal or spatial separation.
- In the forced choice discrimination task, the observer does not know which stimulus is the standard. The task of the subject is to indicate which one of the two stimuli possess more fo some quality.
- In the discrimination task with reminder, the observer is aware of which stimulus is the standard and the task is to say if the comparison felt larger than the standard. This task variant is susceptible to responses biases and should be avoided.
- Note: The distinction between these two variants is not always clear but it has implication on their analysis within the signal detection theory framework (MacMillan & Creelman, chapter 7).

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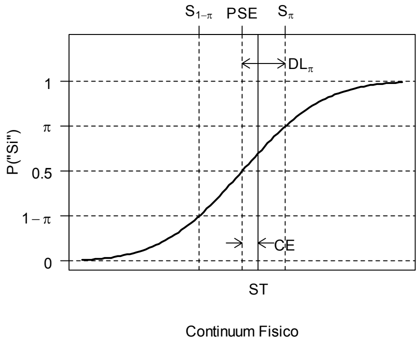
Difference threshold

- In general, the psychometric function relates the probability of judging the comparison stimulus as being larger than the standard to the intensity of the comparison stimulus.



- The **point of subjective equality** (PSE) is the stimulus that elicits 50% of positive responses.
- The **difference threshold** corresponds to the increase of stimulus value DL_π value such that the stimulus S_π = PSE + DL_π elicits π=75% of positive response. Note that this definition depends on the choice of π.

- Alternatively, the difference threshold can be defined as half the distance between the two stimoli $S_{1-\pi}$ and S_π that elicit 1- π = 25% e π=75% of positive responses:

$$DL = (S_\pi - S_{1-\pi})/2$$

The two definitions are equivalent if the psychometric function is symmetric with respect to the PSE.

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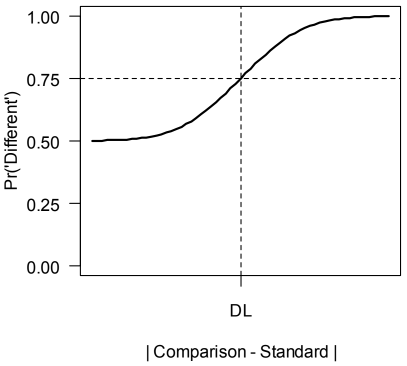
Same-different task

- The **same-different task** is a variant of the forced choice discrimination task where the subject must indicate whether he has perceived a difference between the two stimuli.



- The **psychometric function** relates the proportion of correct responses to the difference between the comparison stimulus. It ranges from 50% (chance level) to 100%.
- The **difference treshold** is defined as the stimulus difference that eleicity a given proportion (e.g., 75%) of the “different” response.

- In many instances, the difference between standard and comparison stimuli has a sign (e.g. the comparison can be either smaller or larger than the standard). One can decides to use only comparison stimuli larger than the standard or the sign of the difference can be randomized across trials. In the latter case, the psychometric function should relate the absolute value of the difference to the probability of perceiving a differenc. Note that this procedure might present some problems in presence of assymetries (e.g., a difference is perceived more easily when the comparison is smaller than the standard than when the opposite true).

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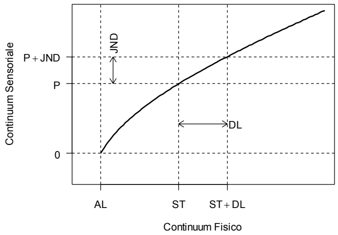
The Weber ratio and Weber's Law



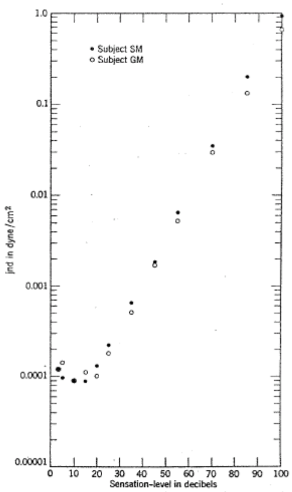


Fig. 4. Intensity discrimination of random noise for two subjects using the quantal method. The jnd is measured in dynes/cm² and is plotted on a logarithmic scale. The stimulus is measured in so-called sensation level units which is decibels *re* subject's threshold (which is approximately 10 db *re* 0.0002 dynes/cm²).

- The difference threshold is closely related to the slope of the psychophysical function. Unless the psychophysical function is linear, its value depends on the intensity of the stimulus.
- The **Weber function f** relates the difference threshold (DL) to the value of the reference stimulus (St)

  DL(St) = f(St)

  In general, the difference threshold increases with the stimulus
- The **Weber ratio K** is the ratio between the difference threshold and the standard

  K(St) = DL(St) / St
- **Weber's Law** is the postulate that Weber ratio is constant

  K = DL(St) / St

Modified Weber Law

- Weber's ratio typically increases markedly when the stimulus becomes weaker.

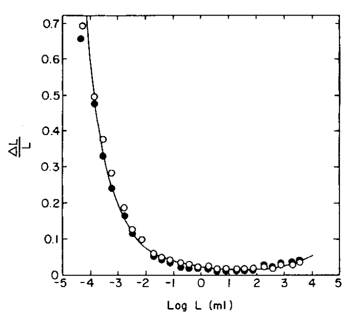


FIG. 1.3. Relation between Δϕ/ϕ and log luminance as shown by Kö (open circles) and Brodhun (solid circles). (From König & Brodhun, 1 after Hecht, 1934, Fig. 27, p. 769.)

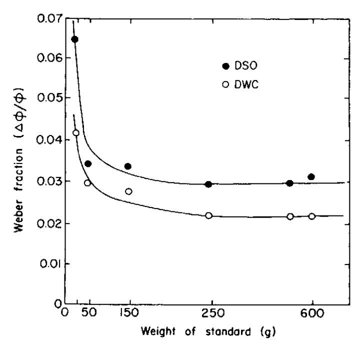
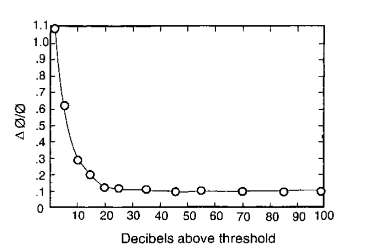


FIG. 1.2. The Weber fraction for lifted weights. The value of Δϕ/ϕ for each of two observers was nearly constant over the stimulus range, except for the lowest stimulus values. (From Engen, 1971.)



1.4. Relation between Δϕ/ϕ and the intensity of auditory noise essed as decibels above the absolute threshold. (From Miller, 1947.)

- A **modified Weber's Law** can account for this observation

$$K = \frac{DL(S)}{a + S}$$

where a is a free parameter.

Psychophysical methods

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Psychophysical methods

- Computation of a sensory threshold implies the presentation of a series of stimuli near in the vicinity of the threshold. The following methods define *the order in which to present the stimuli*. They might be adapted to the various previously defined tasks to measure the absolute or difference threshold.

- Several ***classical methods*** dating back to the very beginning of Pyschophysics are still in use today
  - Method of constant stimuli
  - Method of limits
  - Method of adjustement

- ***Adaptive methods*** use the previous response(s) of the observer to fixe the next stimulus.
  - Staircase methods
  - Modern adaptive methods (PEST, QUEST, etc.) select the stimuli in some statistically optimal fashion.

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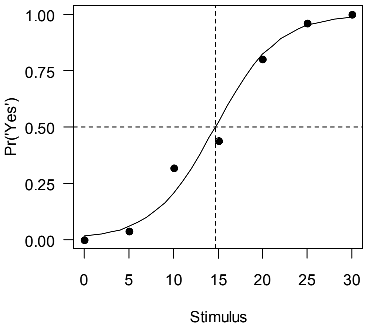
Method of constant stimuli

S	P('Yes'|S)
0	0.00
5	0.04
10	0.32
15	0.44
20	0.80
25	0.96
30	1.00



- The method of constant stimuli is the procedure of repeatedly using the same set of M stimuli. Typically, the set contains between 5 and 9 stimuli, which are presented between 10 and 100 times each (20 is a common value).
- The method of constant stimuli can be used with detection and discrimination tasks. For each stimulus value, the proportion of positive responses is computed (e.g., proportion of "yes" response in a yes-no detection task or proportion of "different" responses in a same-different discrimination task)
- The method of constant stimuli provide all the information necessary to fit a psychometric function and compute the absolute or difference tresholds.

Issues with the method of constant stimuli

- *Advantages:*
  - If the the stimuli are well selected and the number of repetitions large enough, the method of constant stimuli provides a complete and precise view of the psychometric function.
  - Under the same conditions, this method yields unbiased and reliable threshold estimates.
  - Easy to administer.
- *Disadvantages:*
  - For this method to be efficient, the stimuli must corresponds to the interval where the psychometric function increases from 0 to 1. This method requires that the experimenter has some knowledge of this range of values before before the experiment. Still, when there is a lot of variability between subject, the methods can still be quite inefficient as many trials are waster over stimulus values away from the threshold.
  - The method of constant stimuli is less efficient than adaptative methods to compute thresholds (Watson & Fitzhug, 1990). Still, there is some value in using a method that gives the whole psychometric function and pilot experiments a faster methods such as the method of limits can be executed to fix the testing interval.
  - *Reference:* Watson & Fitzhugh (1990) The method of constant stimuli is inefficient. Perception & Psychophysics, 47(1):87-91.

The methods of limits

Su	1	1	2	2	3	3	1	1	2	2	3	3
	D	A	D	A	D	A	D	A	D	A	D	A
91					P							
89			P		Yes							
87	P		Yes		Yes						P	
85	Yes		Yes		Yes		P				Yes	
83	Yes		Yes		Yes		Yes		P		Yes	
81	Yes		Yes		Yes		Yes		Yes		Yes	
79	Yes	Yes	Yes		Yes	Yes	Yes		Yes	Yes	Yes	
77	No	No	No	Yes	Yes	No	Yes	Yes	Yes	No	No	Yes
75		No		No	No	No	Yes	No	Yes	No		No
73		No		No		No	No	No	No	No		No
71		No		No		No		No		No		No
69		No		No		No		No		No		No
67		No		No		No		No		No		P
65		P		No		No		P		No		
63				No		P				No		
61				P						No		
59										P		
T_D	78		78		76		74		74		78	
T_A		78		76		78		76		78		76

Franco Purghé, Tabella 2.1, p. 57

$$T_D = \frac{78+78+76+74+74+78}{6} = 76.33$$

$$T_A = \frac{78+76+78+76+78+76}{6} = 77$$

$$AL = \frac{76.33+77}{2} = 76.67$$

- The experimenter starts by presenting a stimulus well above or well below threshold; on each successive presentation the threshold is approaced by changing the stimulus intentisity by small amount until the response of the subject senses. Typically, the experimenter alternate between the ascending and descending series.
- For each series, a instantaneous threshold is computed (mid-distance btween the two last stimuli). The threshold is the average value of all

Issues with the methods of limits

- *Advantages:*
  - The methods of limits is a simple and efficient procedure for determining sensory thresholds.
  - Often used in clinics (e.g., audiometry or oculometry)
- *Disadvantages:*
  - **Errors of habituation** correspond to a tendency to repeat the same answer even after the sensation has changed. The effect is to increase the threshold in ascending series and to decrease it in descending ones.
  - **Error of expectations** corresponds to an anticipation of the change of sensation. The effect of errors of expectation on thre thresholds is opposite to the effect of errors of habituation.
  - The choice of the **initial value** of the series can also have an effect. For this reason, initial values are varied between repetation.

Staircase method

The Staircase Method (=Up-Down rule, Dixon and Mood, 1948):

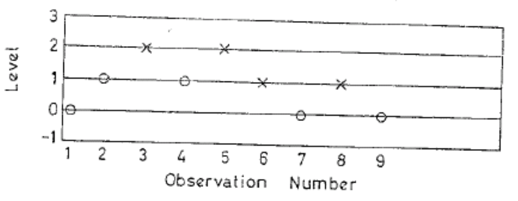


FIGURE 2. Typical pattern of results from the UD rule.

- First observation made at best guess available for PSE [$F^{-1}(0.5)$].
- When a positive response (*cross* in Figure) is obtained, the following observation is taken at the next lower leve, and when a negative response (*circle*) is obtained, the following observation is taken at the next higher level.
- The procedure is terminated after:
  - a fixed number of observations
  - a fixed number of reversals (or runs).

- Computation of absolute threshold or PSE [$F^{-1}(0.5)$]:
  - Average of final observations (e.g., all observations after first run).
  - Average of peak and valleys (example: (2+1+2+0+1+0)/6=1) or average of last *n* peak and valleys.
  - Fitting a psychometric function using maximum likelihood
- Computation of difference threhsold
  - Standard deviation of peaks and valleys migh be viewed an estimate of σ_{PSE}
  - Fitting of a psychometric function

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Issues with the staircase method

- *Advantages*
  - Main advantage of staircase method is its simplicity and enonomicity in terms of number of trials necessary to obtain an estimate of the threshold.
- *Issues*
  - **Selection of initial value**: It is difficult to have a good estimate for the initial value of the staircase method. If the estimate is not good, there is often a bias in the estimated PSE in the direction of the initial value.
  - **Selection of the step value**: It is important that the subject does not perceive the link existing between his response and the choice of the next stimulus in the staircase method. Ideally, the subject should not be able to perceive the direction of the sequence. This imposes the use of small steps. On the other hand, small steps make the experiment long since it might be necessary to present more stimuli to see a reversal. Finding the ideal step is not obvious.
  - **Systematicity**. The subject might count the number of positive and negative responses to artificially stabilize the sequence around some value.
  - Using the standard deviation of reversal points is not a reliable way to estimate the slope of the psychometric function.

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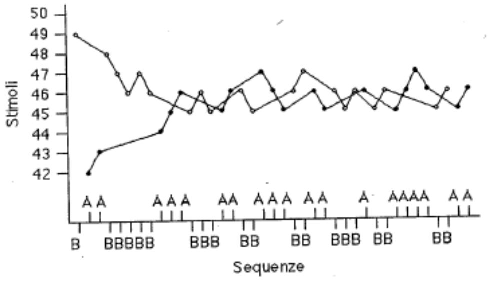
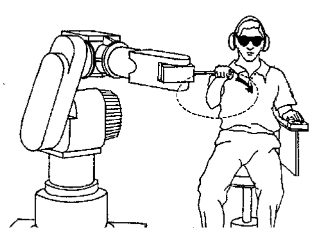
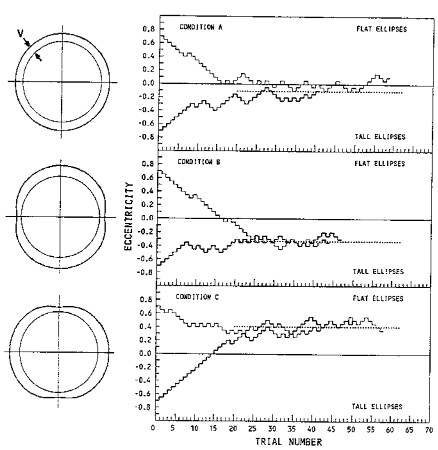
Double staircase method



- Interleaved presentation of two sequences (A e B) of stimulus (see Figure). L'alternance systematic (ABAB...) is not to be recommended.
- Since the sequence are randomly alternated, the subject cannot perceive any logical order in the presentation of the stimuli. This make it impossible to detect, for example, the link existing the reponse and the selection of the next stimulus.
- The threshold is estimated by computing the mean stimulus intensity values corresponding to the last *n* peaks and valleys (or to the average of their intermediary points).
- One sequence has a starting point clearly above (sequence B) and the other sequence clearly below (sequence A) the threshold.
- Possible to test for an effect of the sequence by comparing last *n* peak and valleys values of each sequence (using a t test).
- To accelerate the convergence of the sequences, it is possible to use a larger step at the beginning of the sequences and to decreases it after after, for example, the first reversal.
- For more info: Cornsweet (1962) The Staircase-Method in psychophysics. Amer. J. Psychol., 75:485-491.

Double staircase example





Viviani, Baud-Bovy, Redolfi (1997) Perceiving and tracking kinesthetic stimuli: Further Evidence of Motor-Perceptual Interactions. J. Exp. Psychol.: Hum. Percept. and Perf., 25(4):1232-1252.

- In Experiment 1, a robot was used to move the hand of the blindfolded subject along a predefined trajectory. The shape could vary from an horizontally oriented ellipse (eccentricity<0) to a vertically oriented ellipse (eccentricity>0). The circle correspond to an ellipse with zero eccentricity.
- On each trial, the subject indicated wether he or she perceived an horizontally or vertically oriented ellipse.The eccentricy of the ellipse presented to the subject was changed in function of the response of the subject according to the UD rule: When the subject responded "vertically oriented ellipse", the eccentricity was decreased and vice-versa. The two staircases were interleaved. Initial values corresponded to an easily perceived vertical (ecc=0.7) and horizontal (ecc=-0.7) ellipses respectively. The temination criterion was at least 16 reversals for both staircases. The horizontal dashed line represented the average values for the last 10 reversal and denotes the stimulus judged as circular.
- In Condition A, which corresponds to a constant velocity profile, the subject responded casually when stimulus was circular (eccentricity close to zero). In conditions B and C, the velocity profile was modulated as it would be with an horizontally or vertically-oriented ellipse (eccentricity). In these conditions, the shape perceived as circular was either and horizontally oriented ellipse (condition B) or a vertically oriented ellipse (condition C).

The Up-Down Transformed Rule

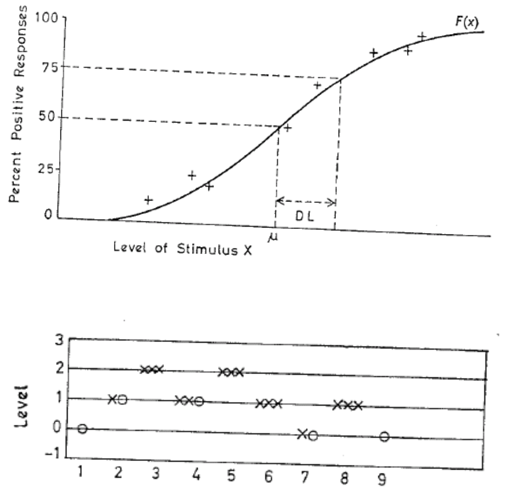


FIGURE 3. Typical pattern of results from the UDTR rule.

Wetherill & Levitt (1965) Sequential estimation of points on a psychometric function. The British Journal of Mathematical and Statistical Psychology, 18(1):1-10.

- The staircase method (or UD rule) yields the intensity of the stimulus that corresponds to 50% of positive responses. The objective of the UDTR rule is to estimate the intensity of the stimulus that yield a probability *P* of positive responses different from 0.5..
- The UDTR Rule: Move intensity of the stimulus down the after D positive responses (*cross* in Figure) and move up after U negative responses (Figure shows an example with 3 positives and 1 negative)
- The first observation should be made at best guess availabe for the desired level [F-1(p)] (see p. 6 of the article for more details on the starting scheme).
- The value of the stimulus that correspond to the desired percentage point *P* is obtained by average the peaks and valley ("Wetherill estimate").

The Up-Down Transformed Rule

TABLE 1. SOME POSSIBLE UDTR RULE PATTERNS

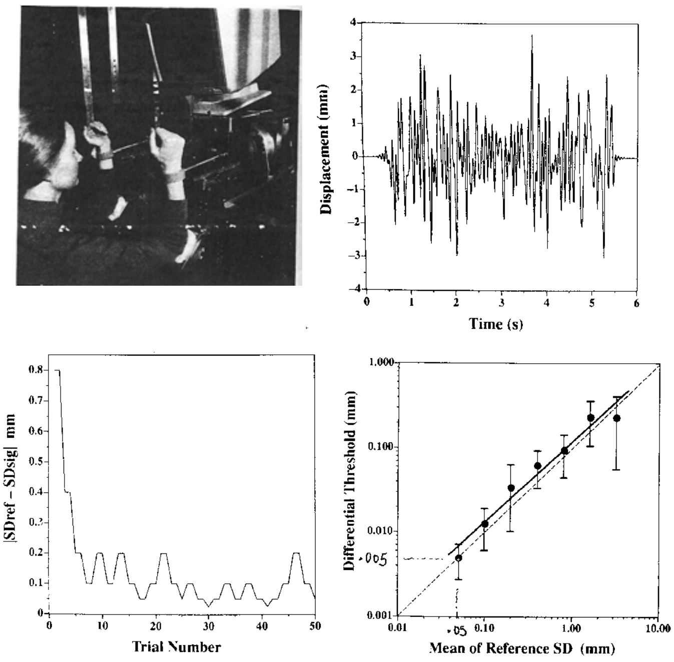
Entry No.	Response Type D	Response Type U	Transformation	Percentage point estimated
1	XX	O,XO	$P=F^2$	0·7071
2	XXX	O,XO,XXO	$P=F^3$	0·7940
3	XXX,XXOX	O,XO,XXOO	$P=F^3(2-P)$	0·7336
4	XXXX	O,XO, etc.	$P=F^4$	0·8409
5	XXXXX	O,XO, etc.	$P=F^5$	0·8705
6	XXXXXX	O,XO, etc.	$P=F^6$	0·8908
7	X,OX	OO	$P=1-(1-F)^2$	0·2929
8	X,OX,OOX	OOO	$P=1-(1-F)^3$	0·2060

etc. by symmetry from entries 1 to 6.

X = positive response. O = negative response.

- The point estimated on the psychometric function will depend on the number of successive positive and negative responses used in the UPTR before changing the value of the stimulus. For example, the Wetherill estimate computed using the UPTR with D=3 and U=1 correspond to 79.4 of correct responses.
- The Table show the percentage point estimated for various UPTRs.

Example of UDTR



Jones et a. (1992) Differential thresholds for limb movement measured using adaptive techniques. Perception & Psychophysics, 52(5):529-535.

- stimuli:small random movements of the arm with a mean of zero and a standard devation of sigma.
- standard: SD_{ref} = 0.05, 0.1, 0.2, 0.4, 0.8, 1.6 and 3.2 mm. Order of presentation randomized across subjects.
- Standard remained fixed for 50 trials (experimental condition). Initial value was of comparison stimulus (SD_{sig}) was initially set to twice that of the standard. SD_{sig} was varied from trial to trial according to a *transformed up-down procedure* aiming at identifying 71% of correct responses: Difference between SD_{ref} and SD_{sig} is *halfed* after two corrected responses and *doubled* after one incorrect one.
- Standard and comparions stimuli were simultaneously presented to the left and right arm (standard was randomly assigned to left or right arm on each trial).
- Differential threshod: *geometric mean* of the absolute difference between standard and comparison for the last 30 trials.

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Issues with transformed staircase methods

- A restriction of transformed up-down staircase methods is that they can estimated only limited number of target levels. To address this issue, Kaernbach (1991) described a variant of the simple up and down procedure where different step sizes are used for the two directions.Unlike the number of positive and negative responses, step sizes can be adjusted to estimate any target probability level.
  - Reference: Kaernbach C (1991) Simple adaptive testing with the weighted up-down method. Perception & Psychophysics, 49:227-229.
- Choosing fixed step size can present difficulty. Intuitively, step size should be larger at the beginning smaller at the end.
- There might be a bias due to the initial starting point.
- Study of efficiency of staircase procedures:
  - García-Pérez MA (1998) Forced-choice staircases with fixed step sizes: asymptotic and small-sample properties. Vision Research, 38(12):38:1861-1881

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Modern adaptive methods

- "Modern" adaptive methods aim at selecting the value of the next stimulus in an optimal fashion, i.e. by finding the value of the stimulus that will bring most information.
- The following methods have used diverse theoretical arguments to select the value of the next stimulus in an optimal fashion and to determine the value of the threshold with the minimum number of trial possible (efficiency).
- Example of methods:
  - PEST: Parametric Estimation by Sequential Testing (Taylor & Creelman, 1967)
  - APE: Adaptive Probit Estimation (Watt & Andrews)
  - QUEST (Watson & Pelli. 1983)
  - ZEST (King-Smith *et al.*, 1994)
- Review:
  - Leek MR (2001) Adaptive procedures in psychophysical research. Perception & Psychophysics, 63(8):1279-1292.

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PEST: Parametric Estimation by Sequential Testing

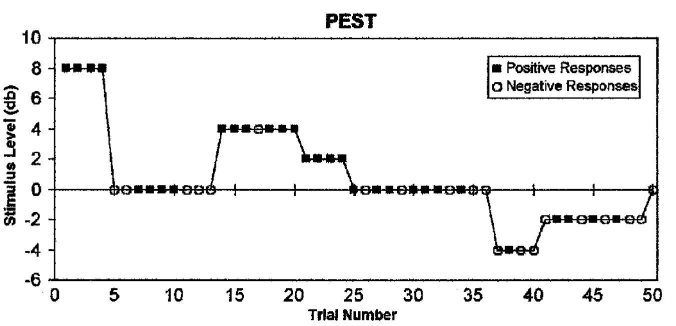

Figure 2. Adaptive track following the PEST procedure. These decibel values are relative to an arbitrary threshold of 0 dB, shown with the horizontal line. These data are modified from Figure 1 in Hall (1981).

References:

Taylor MM,Creelman CD (1967) PEST: Efficient estimates on probability functions. Journal of the Accoustical Society of America, 41:782-787.

Harvey LO (1997) Efficient estimation of sensory threshold with ML-PEST [description of a set of C/C++ routines implementing a variant of PEST].

- Method based on the presentation of a block of trials at the same intensity level. The intensity level is changed when the number of correct responses deviates significantly form that expected from the threshold-criterion probability.
- A new block is presented at a lower or higher intensity depending on the wether the observed probability was above or below the criterion.

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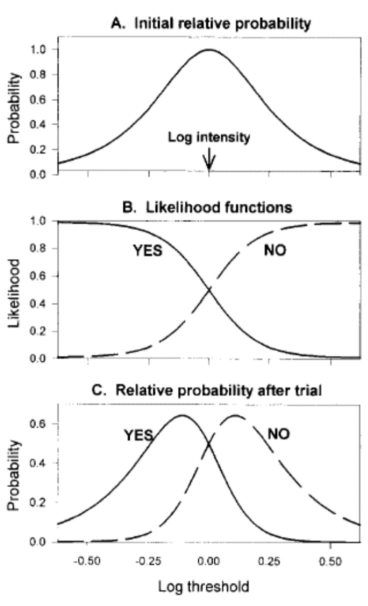
QUEST

- The main idea of the QUEST method is to use Baye's theorem to combine new information (the response of the subject to the presentation of the stimulus) with the previous (prior) knowedlge about the threshold position in order to obtain a new more accurate (posterior) estimate of threshod position.
- QUEST method is commonly used and popular because it has been implemented both in C and Matlab. It is also part of Psychotoolbox, a set of Matlab routines, that have been developped to do psychophysics of the vision and that are used by many groups.
- For more information and software, see the website of Denis Pelli at New-York University: http://www.psych.nyu.edu/pelli/software.html
- If you need it, you can also ask me. I have implemented my own version of QUEST in C and R.
- Reference: Watson AB, Pelli DG (1983) Quest: A Bayesian adaptative psychometric method. Perception & Psychophysics, 33(2):113-120.

QUEST details



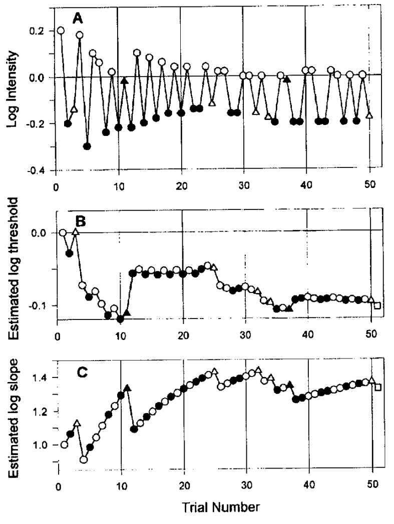
- The initial guess about the location of the treshold is represented by a probabolity density function $q_{prior}(T)$. As an initial guess (before the first trial), Watson and Pelli propose to use a gaussian pdf centered on our best guess about the threshold location with a relatively large standard deviation (since were are not sure yet about the correctness of our guess).
- Baye's theorem is used to update the priod pdf $q_{prior}(T)$ into another pdf $q_{posterior}(T \mid D)$ that incorporates the data D (i.e., the response r_i of the subject to the current stimulus x_i):

$$q_{posterior}(T \mid r_i) = \frac{q(r_i \mid T) q_{prior}(T)}{q(r_i)}$$

- The best estimate for the threshold at any point in time in the experiment is the value that corresponds to the mean[1] of the posterior pdf . This value (or a value nearby) is used as the next stimulus $x_{i+t}=E[q_{posterior}(T \mid D)]$.
- This procedure is repeated for each trial. Note that the posteriod pdf for one trial becomes the prior pdf for the next trial.

[1] In their original paper, Watson & Pelli used the mode instead of the mean. King-Smith et al. 1994 showed that using the mean was more effecient and called their variant ZEST.

Slope estimation with adaptive methods



King-Smith PE, Rose D (1997) Principles of an adaptive method for measuring the slope of the psychometric function. Vision Research, 37(12):1595-1604.

- When estimating the slope parameter, stimuli should be paced at two intensities above and below threshold (King-Smith & Rose, 1997).
- The PEST, QUEST and ZEST methods can be adapated to find the stimulus intensity that correspond to various probabilities of positive responses.
- It also exists methods that can simultaneously estimate the threshold and slope of the psychometric function (remember that the slope is a fixed parameter in the QUEST).
- The figure in the left show the values of the stimuli at the "high" (empty circles) and "low" (solid circles) intensities during the experiment. The threshold and slope are estimated after each response during the experiment (see middle and bottom panels)

Fitting psychometric functions

Data sets

- We assume that only two responses are possible. For example,
  - "Yes" when stimulus is perceived and "No" when the stimulus is not perceived for a detection task.
  - "Stimulus>Standard" and"Stimulus<Standard" for a discrimination task.
- Let y_i be the subject's response to presentation of ith stimulus x_i:

$$y_i = \begin{cases} 1 & \text{if} \quad \text{response is positive} \\ 0 & \text{otherwise} \end{cases}$$

A "positive response" can be "Yes" for a detection task or " Stimulus>Standard" for a discrimination task.

- The data set is the compete list of stimulus and subject's response:

$$\{(x_1,y_1),...,(x_N,y_N)\} = \{(x_i,y_i),\ i=1,..,N\}$$

where N is the total number of presentation

- When a stimulus is presented several times (e.g., as it is the case with the method of constant stimuli), it is possible to compute the proportion of positive response p_i for the i^{th} stimulus:

$$p_i = \frac{1}{n_i}\sum_{i=1}^{n_i} y_i$$

where n_i is the number of repetitions for the i^{th} stimulus

- In this case, the data set is formed by the proportions of positive response and the number of repetition for each stimulus:

$$\{(x_i,\ p_i,\ n_i),\ i=1,..,N_S\}$$

where N_S is the number of stimulus

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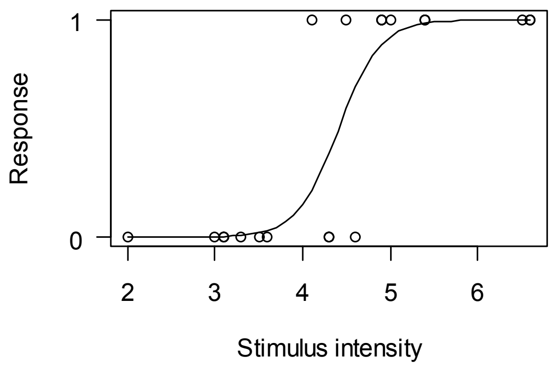
Data set 1 (ungrouped data)

i	x_i	y_i	i	x_i	y_i
1	2.0	0	11	4.5	1
2	3.0	0	12	4.6	0
3	3.1	0	13	4.9	1
4	3.1	0	14	4.9	1
5	3.3	0	15	5.0	1
6	3.5	0	16	5.4	1
7	3.6	0	17	5.4	1
8	4.1	1	18	6.5	1
9	4.3	0	19	6.6	1
10	4.3	0	20	6.6	1



- Results of an experiment with 20 stimulus values x_i and the corresponding subject's response y_i.
- Note that most stimulus value were presented only once.
- Typically, these values will have been obtained using an adaptative methods (e.g., staircase or QUEST method) but the results of any psychophysical method can be presented in this format..
- The plot shows a candidate psychometric function.

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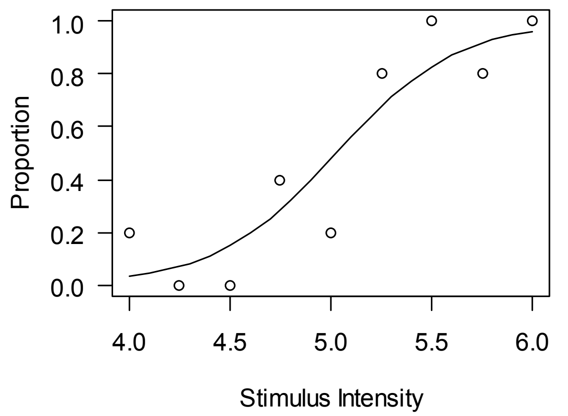
Data set 2 (grouped data)

i	x_i	p_i	n_i
1	4.00	0.2	5
2	4.25	0.0	5
3	4.50	0.0	5
4	4.75	0.4	5
5	5.00	0.2	5
6	5.25	0.8	5
7	5.50	1.0	5
8	5.75	0.8	5
9	6.00	1.0	5



- Results of an experiment with 9 stimulus values x_i and five repetitions per stimulus. The table indicate the proportion p_i of positive responses.
- Typically, these values will have been obtained using the method of constant stimuli.
- The plot shows a candidate psychometric function.
- Note that the data in this table corresponds to 45 subject's responses. For example, for the first stimulus (x_1=4) the subject responded once positively and four times negatively, etc.
- Exercise. Make a table with the all the subject's responses. Why are all the responses multiple of 0.2?

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Psychometric functions

- By definition, the psychometric function $F(x|\theta)$ models the probability π of a positive response

$$\pi = \Pr(Y = 1 \mid x) = F(x \mid \theta)$$

  where θ represents the parameters of the psychometric function.

- Various functions $F(x|\theta)$ have been used to model the probability of positive response (e.g., the logistic, the probit or the Weibull function). Although the analytical form of these functions differ (see next slide), they have many common properties:
  1. They are all S-shaped (sigmoidal) and they all increase monotonically from 0 to 1.
  2. They have parameters θ which needs to be fitted to the data (the fitting problem).

- In the following, we start by describing the properties of these psychometric functions and then we will introduce methods to estimate the parameters of the psychometric function

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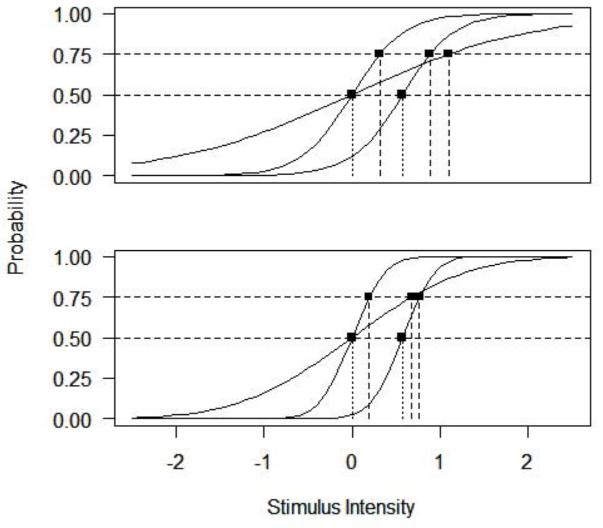
Psychometric functions

- Most theoretical psychometric functions have two parameters α and β that are related to the location (AL or PSE) and slope (DL) of the curve. The precise meaning of these parameters, and their relation to the AL, PSE or DL, depends on the variant of the psychometric function used.



- The logistic function

$$f(x\mid\alpha,\beta)=\frac{1}{1+\exp\left(-(\alpha+\beta x)\right)}$$

- The "probit" function (= integral of a gaussian probability density function)

$$f(x\mid\alpha,\beta)=\int_{-\infty}^{\alpha+\beta x}\frac{1}{\sqrt{2\pi}}e^{-z^2}dz=\Phi(\alpha+\beta x)$$

- The Weibull function (not shown)

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Logistic function

- **Definition** The logisitic function is

$$F(x\mid\alpha,\beta)=\frac{\exp(\alpha+\beta x)}{1+\exp(\alpha+\beta x)}$$

or

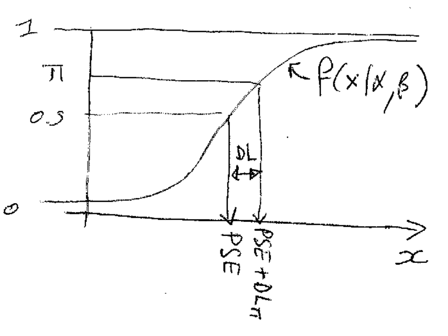
$$F(x\mid\alpha,\beta)=\frac{1}{1+\exp\left(-(\alpha+\beta x)\right)}$$

where α is the intercept and β the slope. Both formulations are completetly equivalent.

- The PSE (or AL) and the DL_π can be recovered from the values of α and β

$$PSE=-\frac{\alpha}{\beta}$$

$$DL_\pi=\frac{1}{\beta}\left(\log\frac{\pi}{1-\pi}\right)$$



- Proof:

$$f(x\mid\alpha,\beta)=\frac{\exp(\alpha+\beta x)}{1+\exp(\alpha+\beta x)}=\frac{\exp(\alpha+\beta x)}{1+\exp(\alpha+\beta x)}\frac{\exp(-(\alpha+\beta x))}{\exp(-(\alpha+\beta x))}$$

$$=\frac{\exp(0)}{\exp(-(\alpha+\beta x))+\exp(0)}=\frac{1}{1+\exp\left(-(\alpha+\beta x)\right)}$$

since $e^x+e^{-x}=e^{x-x}=e^0=1$.

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The logistic regression

- In literature about logistic regression, the logistic model is sometimes presented as linear regression between the so-called logit of the proportions and the predictor variables:

$$\text{logit}(\pi) = \alpha + \beta x$$

where the logit is defined as

$$\text{logit}(\pi) = \log\frac{\pi}{1-\pi}$$

- It can be easily shown that this formulation is stricly equivalent to the previous one.
- This formulation justifies the interpretation of the parameters α and β in terms of odd ratios (see a textbook on logistic regression).

- Proof:

$$\pi = F(x \mid \alpha,\beta) = \frac{1}{1+\exp(-(\alpha+\beta x))}$$
$$\Rightarrow \pi[1+\exp(-(\alpha+\beta x))] = 1$$
$$\Rightarrow \pi \exp(-(\alpha+\beta x)) = 1-\pi$$
$$\Rightarrow \log\pi - (\alpha+\beta x) = \log(1-\pi)$$
$$\Rightarrow \log\pi - \log(1-\pi) = \alpha+\beta x$$
$$\Rightarrow \text{logit}(\pi) = \log\frac{\pi}{1-\pi} = \alpha+\beta x$$

Inverse of the logistic function

- The inverse of the logistic function is

$$x = F^{-1}(\pi \mid \alpha,\beta) = \frac{1}{\beta}\left(\log\frac{\pi}{1-\pi} - \alpha\right)$$

- Proof:

$$\log\frac{\pi}{1-\pi} = \alpha+\beta x$$
$$\Rightarrow x = \frac{1}{\beta}\left(\log\frac{\pi}{1-\pi} - \alpha\right)$$

- The inverse of the logistic function can be used to compute the PSE and DL:

$$PSE = -\frac{\alpha}{\beta}$$
$$DL_\pi = \frac{1}{\beta}\left(\log\frac{\pi}{1-\pi}\right)$$

- Proof

$$PSE = F^{-1}(0.5 \mid \alpha,\beta) = \frac{1}{\beta}\left(\log\frac{0.5}{0.5} - \alpha\right) = -\frac{\alpha}{\beta}$$
$$DL_\pi = F^{-1}(\pi \mid \alpha,\beta) - PSE = \frac{1}{\beta}\left(\log\frac{\pi}{1-\pi} - \alpha\right) - \left(-\frac{\alpha}{\beta}\right)$$
$$= \frac{1}{\beta}\left(\log\frac{\pi}{1-\pi} - \alpha + \alpha\right) = \frac{1}{\beta}\left(\log\frac{\pi}{1-\pi}\right)$$

A variant of the logistic function

- A slighlty different formulation of the the logistic function is

$$F(x \mid \alpha', \beta) = \frac{1}{1 + \exp\left(-\beta(x - \alpha')\right)}$$

- The relationship between α and α' is simple

$$\alpha = -\beta\alpha'$$

$$\alpha' = -\frac{\alpha}{\beta} = PSE$$

- Note that α' can be directly interpreted as the PSE (or AL) with this formulation.

- The value of β and the definition of the DL are the same with both formulations.

- Proof:

$$\beta(x - \alpha') = -\beta\alpha' + \beta x = \alpha + \beta x$$
$$\Rightarrow \alpha = -\beta\alpha'$$
$$\Rightarrow \alpha' = -\frac{\alpha}{\beta}$$

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The probit function

- **Definition** The probit function is the integral of the standardized normal distribution

$$F(x \mid \alpha, \beta) = \int_{-\infty}^{\alpha + \beta x} \frac{1}{\sqrt{2\pi}} e^{-z^2} dz$$

where α is the intercept and β the slope estimated by most softwares.

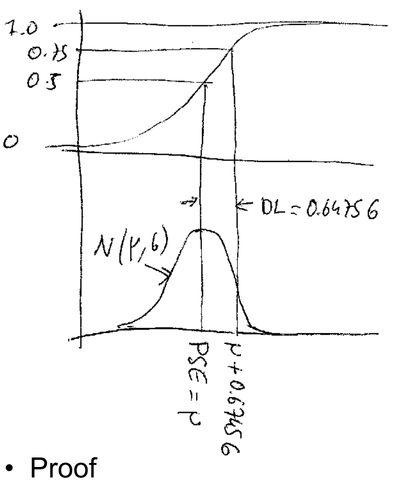
- The mean and standard deviation of underlying normal distribution are

$$\mu = -\alpha / \beta$$
$$\sigma = 1/\beta$$

- The PSE (or AL) and DL_π correspond to:

$$PSE = \mu = -\alpha / \beta$$
$$DL_\pi = u_\pi \sigma = u_\pi / \beta$$

where u_π is the critical values such that $\Pr(Z<u_\pi)=\pi$ for the standardized normal probability distribution (e.g, $u_\pi = 0.6745$ for $\pi=0.75$).



- Proof

$$z = \frac{x - \mu}{\sigma} = \frac{-\mu}{\sigma} + \frac{1}{\sigma} x = \alpha + \beta x$$
$$\Rightarrow \beta = \frac{1}{\sigma} \quad \Rightarrow \sigma = \frac{1}{\beta}$$
$$\Rightarrow \alpha = \frac{-\mu}{\sigma} \quad \Rightarrow \mu = -\alpha\sigma = \frac{-\alpha}{\beta}$$

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A variant of the probit function

- A variant of the probit function is the integral of the normal distribution with mean μ and standard deviation σ :

$$F(x \mid \mu, \sigma) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{\xi-\mu}{\sigma}\right)^2} d\xi$$

- In this case, the parameters are the mean and standard deviation of underlying normal distribution.

- As before, the PSE (or AL) and DL_π are simply PSE $=\mu$ and $DL_\pi = u_\pi \sigma$.

- The probit function cannot be inversed.

Exercise

- Plot the logistic and probit psychometric functions over for various values of α and β:
  - Use values α=0.0 and β=1.0, α=0.0 and β=3.5, α=2.0 and β=3.5, .
  - Plot the values of the psychometric function for range of stimulus intensities from -2.5 to +2.5 spaced by 0.25
  - Make the plots with SPSS (using CDFNORM for the probit function) and/or EXCEL (using NORMSDIST for the probit function)

- For each psychometric function and pair of values α and β, compute the PSE and the DL.

α	β	Logistic		Probit		
		PSE	DL	PSE	σ	DL
0.0	1.0	0	1.099	0	1.000	0.674
0.0	3.5	0	0.314	0	0.286	0.193
-2.0	3.5	0.571	0.314	0.571	0.286	0.193

The fitting problem

- The problem: Given a data set (responses of a subject for each presentation of the stimulus), how do we estimate (compute) the parameters of the psychometric function?

- It exists two principal methods to fit a curve to the data (i.e. to estimate parameters estimate
  1. The minimum-squares method
  2. The maximum likelihood estimation method

- Both methods are extremely general and can be used to fit about any model (including all psychometric functions considered so far) to the data. However, it is more correct to use the maximum-likelihood estimation method when fitting a psychometric function.

Maximum likelihood estimation

- The likelihood function this the probability of observing a given data set (i.e., the subject's responses $\{y_1,...,y_N\}$) for a given the psychometric function.

$$L(\theta) = \Pr(y_1,..., y_N \mid \theta, x_1,..., x_N)$$

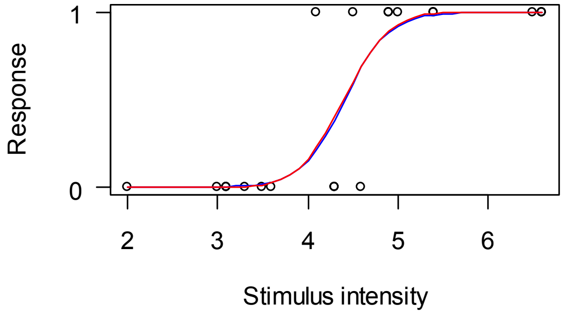
where $\boldsymbol{\theta}$ are the parameters of the psychometric funtion.

- The idea of maximum likelihood estimatation is to find the parameters θ of the psychometric function that maximize the likelihood function, i.e. that maximize the probability of observing the given data set

- Unlike linear regression, there are no analitical formula to compute the values of the parameters $\boldsymbol{\theta}$ that maximize the likelihood function. It is necessary to use an iterative algorithm:
  - These algorithms start with an initial guess that will be iteratively improved.
  - It is important to verify that the algorithm has *converged* toward a stable value.
- Estimates can slightly differ between softwares if different convergence criteria are used.

Exercise.

- Compute the parameters of the logistic and cumulated normal probability functions that best fit the first data set.
- Plot the data and both psychometric functions on the same plot.
- Compute the PSE and $DL_{0.75}$ for both psychometric functions.



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Computing the likelihood function

- By definition, the psychometric function models the probability of a positive response for any stimulus intensity x:

$$\pi = \Pr(Y = 1 \mid \theta, x) = F(x \mid \theta)$$

- If we assume that the parameters of the psychometric functions are known, then it is possible to compute the probability π_i of observing the response y_i for a given stimulus x_i:

$$\Pr(Y = y_i \mid \theta, x_i) = \begin{cases} \pi_i = F(x_i \mid \theta) & if \ y_i = 1 \\ 1 - \pi_i = 1 - F(x_i \mid \theta) & if \ y_i = 0 \end{cases}$$

- Assuming that the responses y_i are independent, then the probability of observing a given set of responses (the data) is:

$$L(\theta) = \Pr(y_1, \ldots, y_N \mid \theta, x_1, \ldots, x_N) = \Pr(y_1 \mid \theta, x_1) \cdots \Pr(y_N \mid \theta, x_N) = \prod_{k \in \{i \mid y_i = 1\}} \pi_k \prod_{k \in \{i \mid y_i = 0\}} (1 - \pi_k)$$

- Maximizing the likelihood function is equivalent to maximizing its logarithm (the so-called log-likelohood function)

$$l(\theta) = \log L(\theta) = \sum_i \left(y_i \log \pi_i + (1 - y_i) \log(1 - \pi_i) \right)$$

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Example

- Let assume a small data set with three observations (two positive and one negative):

  y_1 = 1 for stimulus x_1, y_2 = 0 for stimulus x_2, y_3 = 1 for stimulus x_3.

- The likelihood function is

$$\begin{aligned}
L(\theta) &= \Pr(Y_1 = 1, Y_2 = 0, Y_3 = 1 \mid \theta, x_1, x_2, x_3) \\
&= \Pr(Y_1 = 1 \mid \theta, x_1)\Pr(Y_2 = 0 \mid \theta, x_2)\Pr(Y_3 = 1 \mid \theta, x_3) \\
&= \pi_1(1-\pi_2)\pi_3 = \pi_1\pi_3(1-\pi_2) \\
&= \prod_{\{k|y_k=1\}} \pi_k \prod_{\{k|y_k=0\}} (1-\pi_k)
\end{aligned}$$

  where $\pi_i = F(x_i \mid \theta)$.

- The log-likelihood is

$$\begin{aligned}
l(\theta) &= \sum_i y_i \log \pi_i + \sum_i (1-y_i)\log(1-\pi_i) \\
&= \left(y_1 \log \pi_1 + y_2 \log \pi_2 + y_3 \log \pi_3\right) + \\
&\qquad \left((1-y_1)\log(1-\pi_1) + (1-y_2)\log(1-\pi_2) + (1-y_3)\log(1-\pi_3)\right) \\
&= \log \pi_1 + \log \pi_3 + \log(1-\pi_2) \\
&= \log(\pi_1\pi_3(1-\pi_2))
\end{aligned}$$

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Mathematical details

- Derivation of the fomula for the log-likelihood:

$$\begin{aligned}
\log L(\theta) &= \log\left(\prod_{k\in\{i|y_i=1\}} \pi_k \prod_{k\in\{i|y_i=0\}} (1-\pi_k)\right) \\
&= \sum_{k\in\{i|y_i=0\}} \log \pi_k + \sum_{k\in\{i|y_i=0\}} \log(1-\pi_k) \\
&= \sum_i y_i \log \pi_i + \sum_i (1-y_i)\log(1-\pi_i) \\
&= \sum_i \left(y_i \log \pi_i + (1-y_i)\log(1-\pi_i)\right)
\end{aligned}$$

  using the fact that $1-y_i = 1$ if $y_i=0$ and vice-versa to sum over i.

- A relatively simple expression for the log-likelihood of the logistic function can be obtained by rewriting the previous equation

$$\begin{aligned}
\log L(\theta) &= \sum_i \left(y_i \log \pi_i (1-y_i) + \log(1-\pi_i)\right) \\
&= \sum_i \left(y_i\left(\log \pi_i - \log(1-\pi_i)\right) + \log(1-\pi_i)\right) \\
&= \sum_i \left(y_i \log \frac{\pi_i}{1-\pi_i} + \log(1-\pi_i)\right)
\end{aligned}$$

  Substituting the logistic function yields

$$\log L(\alpha,\beta) = \sum_i \left(y_i(\alpha+\beta x_i) + \log\left(1 - \frac{1}{1+e^{-(\alpha+\beta x_i)}}\right)\right)$$

  which can be simplified to

$$\log L(\alpha,\beta) = \sum_i \left(y_i(\alpha+\beta x_i) - \log\left(1+e^{\alpha+\beta x_i}\right)\right)$$

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Fit psychometric function [MATLAB]

logisitic.m

```
function p=logistic(x,alpha,beta)
    % logistic
    p = 1./(1+exp(-(alpha+beta.*x)));
```

probit.m

```
function p=probit(x,alpha,beta)
    % probit
    x = alpha+beta.*x;
    p=0.5+0.5*erf(x./sqrt(2));
```

loglikelihood.m

```
function ll=loglikelihood(par,fhandle,X,Y)
    % fhandle must refer to a psychometric function that computes
    % the probability of a positive response (e.g. probit or logistic)
    P = feval(fhandle,X,par(1),par(2));
    % log-likelihood function
    ll= -sum(Y.*log(P+eps)+(1-Y).*log(1-P+eps))
```

main.m

```
X=data(:,1); % stimuli
Y=data(:,2); % responses

% fminsearch: minimization loglikelihood with simplex method
options=optimset('fminsearch');

disp('Maximum likelihood fit of various psychometric functions:  ');

[x,fval,exitflag]  = fminsearch(@loglikelihood,[-10 3],options,@probit,X,Y);
disp('Probit: ');
disp(['alpha=' num2str(x(1)) ' beta=' num2str(x(2)) ' log-likleihood=' num2str(-fval)]);

[x,fval,exitflag]  = fminsearch(@loglikelihood,[-20 3],options,@logistic,X,Y);
disp('Logit: ');
disp(['alpha=' num2str(x(1)) ' beta=' num2str(x(2)) ' log-likleihood=' num2str(-fval)]);
```


Fit probit function in R (glm)

Script:

```
# make data set
data<-data.frame(
          x=seq(4,6,0.25),
          p=c(0.2,0.0,0.0,0.4,0.2,0.8,1.0,0.8,1.0),
          n=rep(5,9))

# fit probit function
fit<-glm(p~x,data=data,family=binomial(link = "probit"),weights=data$n)
summary(fit)

# compute PSE and DL
-coef(fit)["(Intercept)"]/coef(fit)["x"]    # 5.0136
prnorm(0.75)/coef(fit)["x"]                 # 0.4252891
```

Ouput:

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  -9.1170     2.2398  -4.071 4.69e-05 ***
x             1.8185     0.4447   4.089 4.33e-05 ***
```

Fit logisitic function in R (glm)

Script:

```
# fit logistic function
fit<-glm(p~x,data=data,family=binomial(link = "logit"),
         weights=data$n)
summary(fit)

# compute PSE and DL
-coef(fit)["(Intercept)"]/coef(fit)["x"]    # 5.026206
log(0.75/(1-0.75))/coef(fit)["x"]           # 0.3320319
```

Output:

```
Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -16.6305     4.5853  -3.627 0.000287 ***
x             3.3088     0.9093   3.639 0.000274 ***
```

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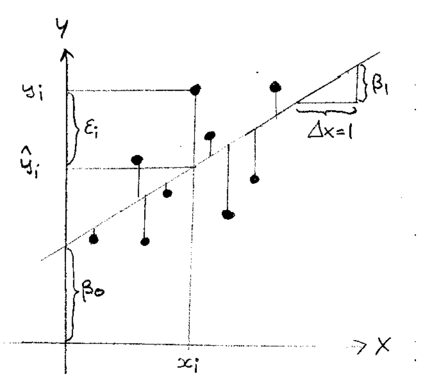
Minimum-squares estimation

- The idea of the method is to find the values of the parameters α and β that minimize the sum of the squares of the distance between the data points y_i and values predicted by the psychometric function:

$$\min_{\alpha,\beta} \sum_i (y_i - F(x_i \mid \alpha,\beta))^2$$



- The method is the same as the one used in linear regression to estimate the parameters of the regression line. However, unlike in the case of the linear regression, there are no formulae to compute directly the values of the parameters of the psychometric function. One needs to use a minimization algorithm that start with initial values for the parameters and then change them in small steps until the "best" values are found.

- General caveats with minimization/maximization algorithm:
  1. It is important to check that the algorithm has *converged* toward stable values (you can increase maximum number of iterations if not)
  2. The best values are not always found (problem of local minima)

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Exercises

- Repeat the same analysis using the other formula of the logistic function:

$$f(x\,|\,\alpha,\beta)=\frac{1}{1+\exp\left(-(\alpha+\beta x)\right)}$$

- Recompute PSE and DL from the values of α and β estimated with this variant of the logistic function
- Check that that psychometric function $f(x|\alpha, \beta)=0.75$ when $x = \mathrm{PSE}+\mathrm{DL}_{0.75}$.
- Estimate the parameter of the probit function. Hint: you need to use CDFNORM(alpha+beta*x) in *Model Expression* and you should also change the initial value of α to -15.
  - Results:

Parameter	Estimate	Std. Error	Lower	Upper
ALPHA	-13.87223409	5.068203621	-25.85663129	-1.887836900
BETA	2.742900994	1.000819205	.376339630	5.109462358

- Compute the PSE and $\mathrm{DL}_{0.75}$ from the results of the probit analysis.
  - Results: PSE=5.0575 and $\mathrm{DL}_{0.75}$ = 0.2459

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MATLAB (nlinfit)

- **Script**

```
% Define data set
x = 4:0.25:6;
y = [0.2 0.0 0.0 0.4 0.2 0.8 1.0 0.8 1.0];

% Define logistic function (note: beta(1) is alpha)
f = inline('1./(1+exp(-beta(2).*(x-beta(1))))','beta','x');

% Nonlinear least-squares data fitting by the Gauss-Newton method
[beta,r,J] = nlinfit(x,y,f,[5 3]);

% 95% interval of confidence
ci = nlparci(beta,r,J);
```

- **Output**

```
>> beta
beta =
        5.0638
        4.5947
>> ci
ci =
        4.8430      5.2846
        0.4814      8.7081
```

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R (nls)

- **Script**

```
# define data set
data<-data.frame(
         x=seq(4,6,0.25),
         p=c(0.2,0.0,0.0,0.4,0.2,0.8,1.0,0.8,1.0),
         n=rep(5,9))
# fit logistic function with Gauss-Newton algorithm
fit<-nls(p~1/(1+exp(-beta*(x-alpha))),
         data = data,
         start= list(beta=3,alpha=5))
summary(fit)
```

- **Output**

```
Formula: p ~ 1/(1 + exp(-beta * (x - alpha)))
Parameters:
      Estimate Std. Error t value Pr(>|t|)
beta   4.59542    1.73992   2.641   0.0334 *
alpha  5.06383    0.09336  54.237  1.9e-10 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.1634 on 7 degrees of freedom
```

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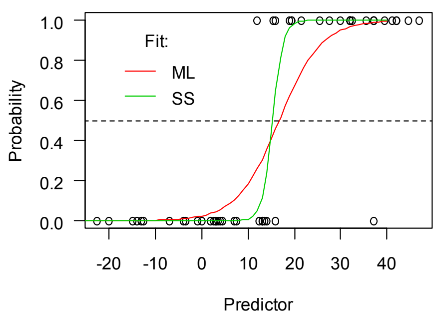
Fitting methods

- The two methods can give quite different results

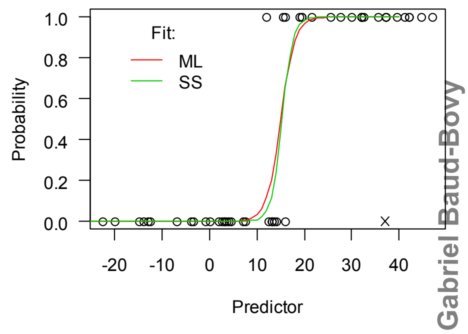


Parameter	Fitting method	
	ML	SS
Alpha	16.703	15.314
Beta	0.221	0.893
deviance	**25.54**	(50.66)
SS	(3.37)	2.77

Fitted function:

$$y = \frac{1}{1+\exp(-\beta(x-\alpha))}$$

- ML fit has a smaller slope because binomial error give more weight to inconsistent responses than SS method
- Both fitting methods give similar results when inconsistent responses (cross) are removed



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