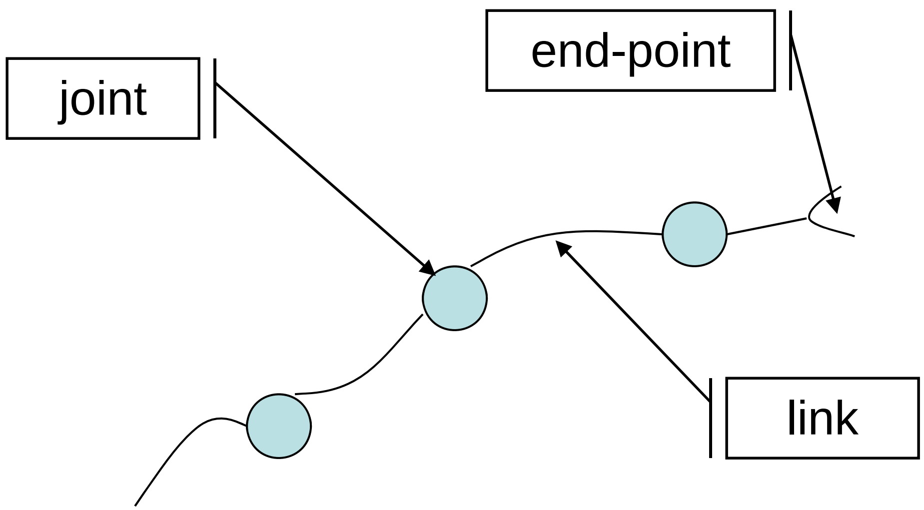
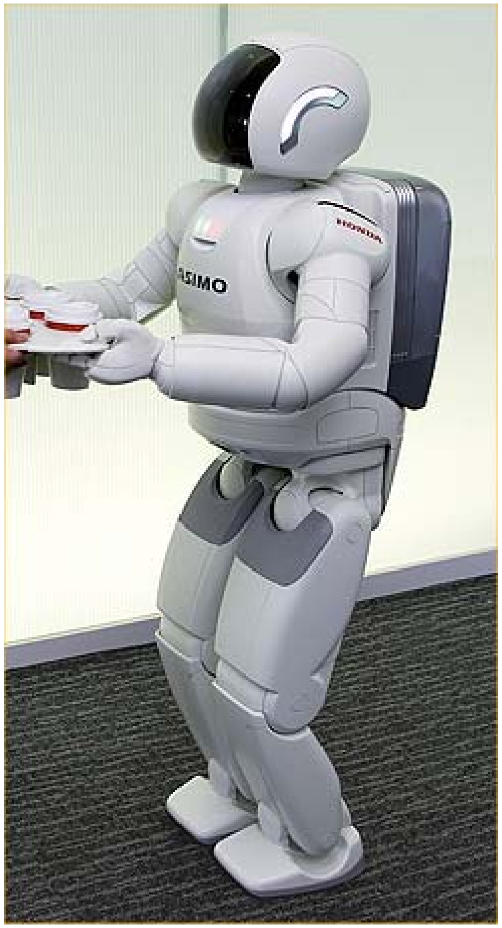
Mechanical systems

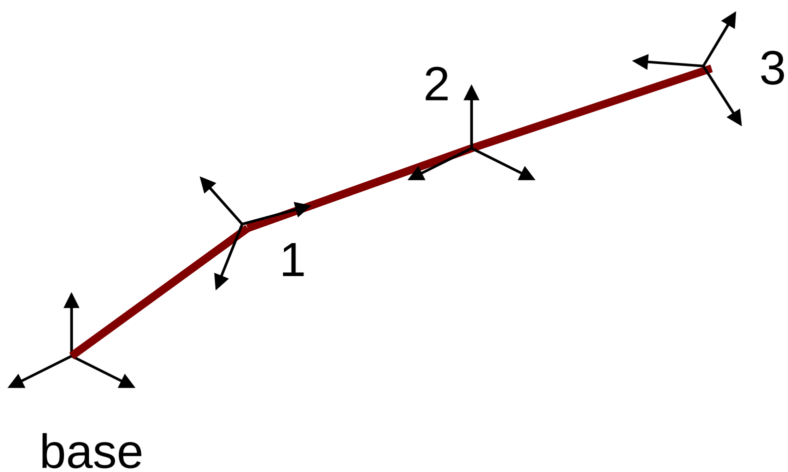
- Things we'd like to model with the help of some trivial physics (but complex notation)





How to describe things mathematically

- One reference frame per link
  - Will be required soon…



Studying what?

	No forces	Forces
No motion	Styling	Static
Motion	Kinematics	Dynamics

Rigid body transformations

$$\|p(t)-q(t)\| = \|p(0)-q(0)\| = \text{constant}$$

- Given that the object is:

$$O \subset \mathbb{R}^3$$

- The motion of the body is represented by a family of mappings:

$$g(t) : O \rightarrow \mathbb{R}^3$$

- A rigid displacement of the body is:

$$g : O \rightarrow \mathbb{R}^3$$

Action on points and vectors

$$g_*(v) = g(q) - g(p)$$

Where:

$$v = q - p$$

Note the difference between points and vectors (although both are represented as 3-tuples of numbers). A vector has magnitude and direction and doesn't belong to a body (free vector).

Then…

$$g : \mathbb{R}^3 \to \mathbb{R}^3$$

is a rigid body transformation if:

$$\|g(p) - g(q)\| = \|p - q\| \text{ for all points } p, q \in \mathbb{R}^3$$

Length is preserved

$$g_*(v \times w) = g_*(v) \times g_*(w) \text{ for all vectors } v, w \in \mathbb{R}^3$$

The cross product is preserved

$\downarrow$

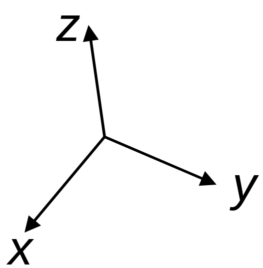
The inner product is also preserved, thus:

$$v^T w = g_*(v)^T g_*(w)$$

I.e. orthogonal vectors remain orthogonal

Some more requirements

- Right handed coordinate systems:



$$z = x \times y$$

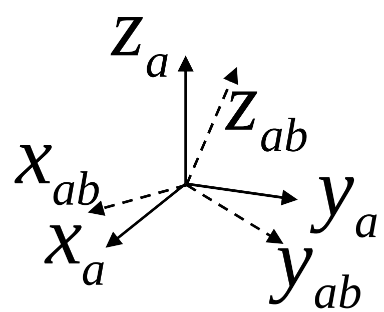
- If a coordinate system is attached to a rigid body undergoing rigid motion:

v_1, v_2, v_3 attached in p then by effect of g

$g_*(v_1), g_*(v_2), g_*(v_3)$ are attached in g(p)

Rotation matrix

$$R_{ab} = [x_{ab} \mid y_{ab} \mid z_{ab}]$$



x_{ab} Coordinates of the B's principal axis *x* relative to A
A is the <u>inertial</u> frame, B is the <u>body</u> frame

$$R_{ab} \in \mathbb{R}^{3\times3}, x_{ab}, y_{ab}, z_{ab} \in \mathbb{R}^{3}$$

Then:

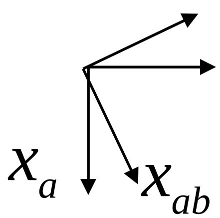
$$x_{ab} y_{ab} = 0 \text{ and so forth...}$$

$$RR^T = R^T R = I$$

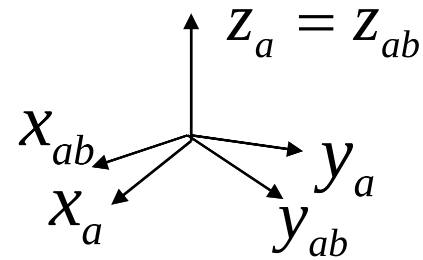
$$\det R = 1 \text{ for right-handed coordinate systems}$$

Rotation matrix (planar case)

Example: rotation along the Z axis $\begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$



$$x_{ab} = \begin{bmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{bmatrix}$$



The group of rotations SO(3)

- The set of 3x3 matrices with these properties is denoted:

$SO(3)$ which means Special Orthogonal of size 3

- That is:

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} : \underbrace{RR^T = I}_{\text{Orthogonal}}, \underbrace{\det R = +1}_{\text{Special}}\}$$

SO(3) is a group under matrix multiplication

1. Closure

$$R_1, R_2 \in SO(3) \Rightarrow R_1 R_2 \in SO(3)$$

2. Identity

$$I \text{ is the identity element } IR = R \;\; \forall R$$

3. Inverse

$$RR^T = R^T R = I, R^T \in SO(3)$$

4. Associativity

$$(R_1 R_2) R_3 = R_1 (R_2 R_3)$$

SO(3) matrices

1. Serve as a representation of the configuration of a rigid body wrt an inertial frame of reference ($R(t)$ is a curve in $SO(3)$).

2. Serve as transformation to map points from one frame of reference to another (see next).

Let's examine point 2.

$q_b = (x_b, y_b, z_b)$ These are projections of *q* on B's axes

$q_a = x_{ab}x_b + y_{ab}y_b + z_{ab}z_b$ With respect to A

$$q_a = [x_{ab} \mid y_{ab} \mid z_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = R_{ab} q_b$$

As a matter of notation R_{ab} maps points from B to A

Vectors

$$R_{ab}v_b = R_{ab}q_b - R_{ab}p_b = q_a - p_a = v_a$$

Rotations are well-defined for vectors

Cross product

$$a \times b = (a)^\wedge b$$

where

$$(a)^\wedge = \hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

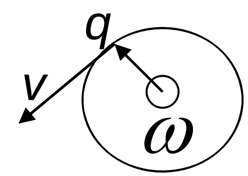
$$R(a \times b) = Ra \times Rb$$

$$R(a)^\wedge R^T = (Ra)^\wedge$$

Proof by direct calculation (i.e. note that they are only 3x3 matrices)

Exponential coordinates

$$\dot{q}(t) = \omega \times q(t) = \hat{\omega} q(t) \qquad \|\omega\| = 1$$



whose solution is:

$$q(t) = e^{\hat{\omega} t} q(0)$$

$$e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \ldots$$ Matrix exponential

which is equivalent to:

$$R(\omega, \vartheta) = e^{\hat{\omega} \vartheta}$$

More on the exp map…

$\hat{\omega} : \hat{\omega}^T = -\hat{\omega}$ Skew-symmetric by definition

$\hat{\omega} \in so(3)$ Which is a vector space over the real number of all skew-symmetric 3x3 matrices

so(3) can be identified with $\mathbb{R}^3$

Then, let’s consider:

$$\hat{\omega} \in so(3), \|\omega\| = 1, \vartheta \in \mathbb{R}$$

We wish to study:

$$e^{\hat{\omega}\vartheta}$$

The exp map (how to compute it)

$$\boxed{e^{\hat{\omega}\vartheta} = I + \hat{\omega}\sin\vartheta + \hat{\omega}^2(1-\cos\vartheta)}$$

Exponentials of skew matrices are orthogonal

i.e. $$e^{\hat{\omega}\vartheta} \in SO(3)$$

In fact:

$$(e^{\hat{\omega}\vartheta})^{-1} = e^{-\hat{\omega}\vartheta} = e^{\hat{\omega}^T\vartheta} = (e^{\hat{\omega}\vartheta})^T \Rightarrow R^{-1} = R^T$$

And:

$$\det(e^{\hat{\omega}\vartheta}e^{-\hat{\omega}\vartheta}) = \det(e^{\hat{\omega}\vartheta})\det(e^{-\hat{\omega}\vartheta}) = \det I = 1$$

$$\det(R^{-1}) = \det(R^T) = \det(R)$$

And by continuity of **det** wrt the elements of *R* and the fact that: $\det\exp(0) = 1$

$$\det e^{\hat{\omega}\vartheta} = 1$$

exp is surjective (many to one)

$$\boxed{\omega \in \mathbb{R} \Rightarrow \hat{\omega} \in so(3), \vartheta \in \mathbb{R} \Rightarrow e^{\hat{\omega}\vartheta} \in SO(3)}$$

There are multiple:

ω, ϑ which map onto a given R

And also:
"any R is equivalent to a rotation around ω by an amount equal to ϑ"

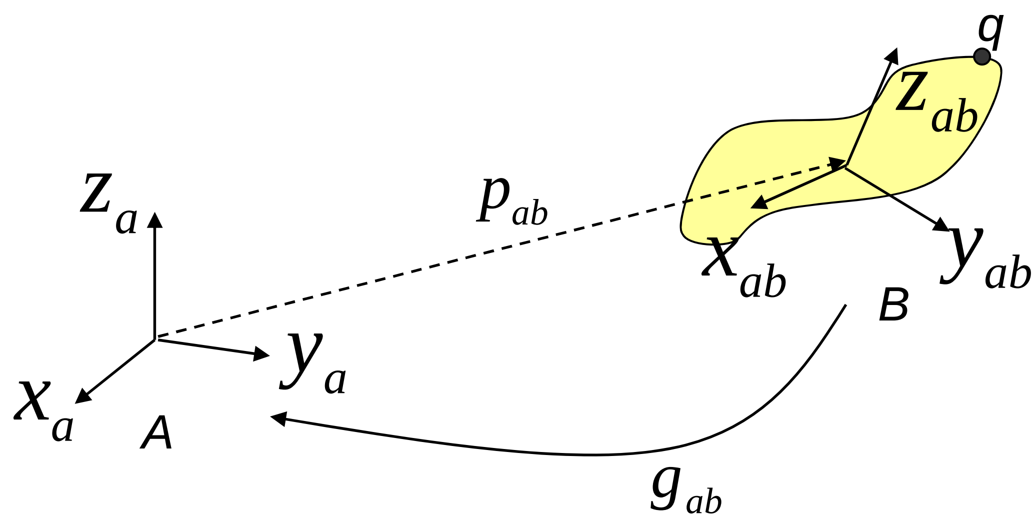
$$\omega \in \mathbb{R}^3, \vartheta \in [0, 2\pi)$$

Other representations of rotations

- Euler angles
- Roll-pitch-yaw
- Quaternions
- etc…

- The exponential representation is called the *canonical* representation or *canonical* coordinates of $SO(3)$

Rotation+translation



To describe the situation we need at least:

$$R \in SO(3),\ p_{ab} \in \mathbb{R}^3$$

Thus we define (Special Euclidean):

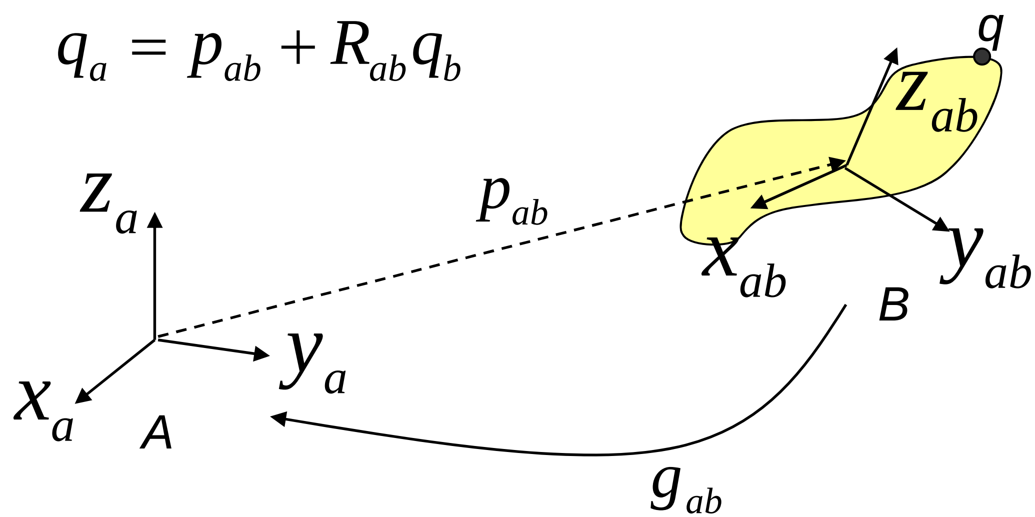
$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3)$$

As before…

- An element of *SE(3)* can serve either as descriptor of a trajectory (configuration) or as the transformation between coordinate frames (A,B in the previous example)

Thus…

$$q_a = p_{ab} + R_{ab} q_b$$



Where:

$$g_{ab} = (p_{ab}, R_{ab}) \in SE(3)$$ Specifies the configuration of *B* wrt *A*

$$g(q) = p + Rq$$ By dropping some of the notation

but simply… $q_a = g(q_b)$

The action on vectors

$$g_*(v) = g(s) - g(r) = p + Rs - p - Rr = R(s - r) = Rv$$

thus a vector is transformed by rotation (only)

Homogeneous representation

- Add a dimension to vectors and points

$$\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \qquad \bar{o} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \longrightarrow \qquad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

Thus, we can sum freely vectors and they'll remain vectors
Sum a point to a vector to get a point (displaced)
Subtract two points to get a vector (rightly so)
While the sum of two points is meaningless (rightly so)

g_{ab} is *affine* in $\mathbb{R}^3$ but becomes linear in $\mathbb{R}^4$

Linear form of *g*

$$\bar{q}_a = \begin{bmatrix} q_a \\ 1 \end{bmatrix} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix} = \bar{g}_{ab} \bar{q}_b$$

$$\forall g_{ab} = (p, R) \in SE(3) \;\; \exists \bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Homogeneous representation of *g* in *SE(3)*

Composition $$\bar{g}_{ac} = \bar{g}_{ab} \bar{g}_{bc} = \begin{bmatrix} R_{ab} R_{bc} & R_{ab} p_{bc} + p_{ab} \\ 0 & 1 \end{bmatrix}$$

C→B→A

Indeed *SE(3)* is a group

1. Composition
2. Identity
3. Inverse

$$\forall g_{ab} = (p, R) \in SE(3) \;\; \exists \bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$g_{ab}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

4. Associativity

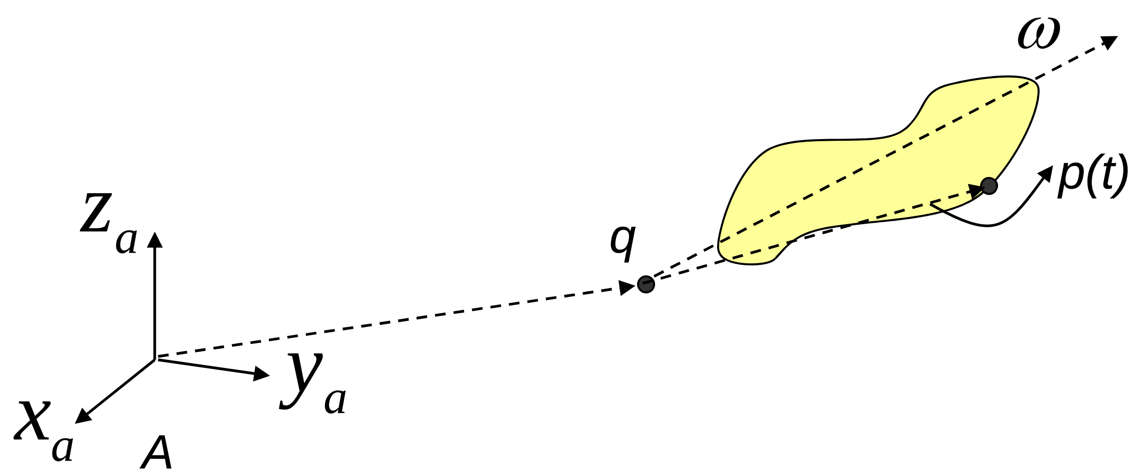
…and

- The same transformation applies to vectors (also in homogeneous coordinates)

$$\bar{g}_*\bar{v} = \bar{g}(\bar{s}) - \bar{g}(\bar{r}) = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

- It can be shown (by calculation) that *g*'s in *SE(3)* are indeed rigid body transformations

Exponential map (twist)



$\dot{p}(t) = \omega \times (p(t) - q)$ as seen earlier $\omega \in \mathbb{R}^3, \|\omega\| = 1, q \in \mathbb{R}^3$

$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$ with $v = -\omega \times q$ Homogeneous coordinates

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \hat{\xi} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow \dot{\bar{p}} = \hat{\xi}\bar{p}$$

…and finally (well sort of)

$$\overline{p}(t) = e^{\hat{\xi}t}\,\overline{p}(0), \text{ with } e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \frac{(\hat{\xi}t)^3}{3!} + \ldots$$

Similarly (for pure translation):

$$\dot{p} = v, \quad \text{thus } \hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

We can define:

$$se(3) = \{(v, \hat{\omega}) : v \in \mathbb{R}^3, \hat{\omega} \in so(3)\} \qquad \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

twist in h coordinates

Which is a vector space

Two (new) operators

$$\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^{6}$$

Called *vee* which recovers the 6D vector that parameterizes the *twist*

$$\begin{bmatrix} v \\ \omega \end{bmatrix}^{\wedge} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

Called *wedge* which builds a homogeneous coordinate representation of a *twist* given the 6 parameters

Exp map from *se(3)* to *SE(3)*

- As before:

mapping

$$\begin{bmatrix} v \\ \omega \end{bmatrix} \in \mathbb{R}^{6} \rightarrow (\hat{\omega}, v) \in se(3) \rightarrow \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \rightarrow \exp \rightarrow g \in SE(3)$$

vector space

homogeneous coordinates, useful!

transformation or trajectory

twist coordinates

The map (explicit)

$$e^{\hat{\xi}\vartheta} = \begin{cases} \begin{bmatrix} I & v\vartheta \\ 0 & 1 \end{bmatrix} & \omega = 0 \\ \begin{bmatrix} e^{\hat{\omega}\vartheta} & (I - e^{\hat{\omega}\vartheta})(\omega \times v) + \omega\omega^T v\vartheta \\ 0 & 1 \end{bmatrix} & \omega \neq 0 \end{cases}$$

Obtained through calculation and a few additional theorems/properties

Interpretation

$p(\vartheta) = e^{\hat{\xi}\vartheta} p(0)$ where $p(\vartheta)$ is the final position

where $p(0)$ is the initial position

and both are specified with respect to the same coordinate frame

$g_{ab}(\vartheta) = e^{\hat{\xi}\vartheta} g_{ab}(0)$ with *g*'s representing the configuration of the rigid body

A *twist* represents relative motion of a rigid body

The exponential map is surjective: many to one