Biorobotics : Lesson N° 3

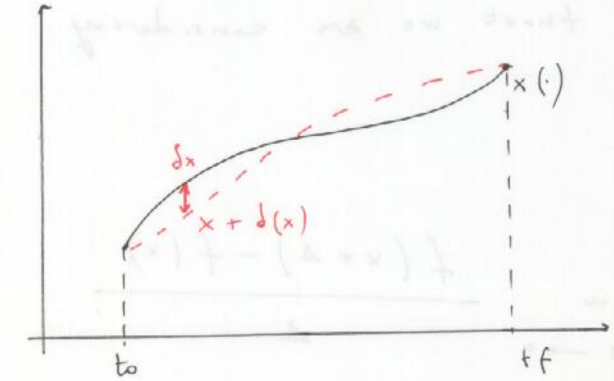
- Book: Optimal control theory
- tools for solving Optimal Control Problems, because most of the best problem consider this kind of control
- $$\min_{x(\cdot)} J(x(\cdot)) = \int_{t_0}^{t_f} L(x(\tau), \dot{x}(\tau), \tau)\, d\tau$$

$x \longrightarrow$ is a trajectory, continuos function with continuos derivative

$$x : [t_0, t_f] \longrightarrow \mathbb{R}$$

- CALCULUS OF VARIATIONS : THEORY
- Def (VARIATION) : Given a function $x : [t_0, t_f] \longrightarrow \mathbb{R}^n$, we call VARIATION of x, the quantity

$$\delta x : [t_0, t_f] \longrightarrow \mathbb{R}^n$$



- Note: Given a function x, the VARIATION δx is a FUNCTION ITSELF

· Def : CONSIDER the INCREMENT of J due to a VARIATION δx of x

$$\Delta J(x+\delta x) = J(x+\delta x) - J(x)$$

If ΔJ can be written as follows:

$$\Delta J(x+\delta x) = \delta J(x, \delta x) + g(x, \delta x) \, \| \delta x \|$$

with $\delta J(x, \delta x)$ LINEAR in δx and

$$\lim_{\|\delta x\| \to 0} g(x, \delta x) = 0$$

$\Longrightarrow$ then the QUANTITY δJ IS VARIATION OF J IN $x(\cdot)$

· Let's suppose that we are considering:

$f(x)$

$$\frac{\partial f}{\partial x} = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$$

the variation of J is basically the same, because:

$$\delta J = \lim_{\|\delta x\| \to 0} \frac{\Delta J - \cancel{g(x, \delta x)} \|\delta x\|}{\|\delta x\|}$$

- Note: we need to define what is the norm of a function $(\|\delta x\|)$

- Def: the norm of $x \; [t_0, t_f] \longrightarrow \mathbb{R}^n$ is a RULE of correspondence that assigns to each function $x : [t_0, t_f] \longrightarrow \mathbb{R}^n$ a REAL number.

  The NORM $\|x\|$ satisfies the following properties:

  (1) $\|x\| > 0$, and $\|x\| = 0$ if and only if $x(t) = 0 \;\; \forall t \; [t_0, t_f]$

  (2) $\|\alpha x\| = |\alpha| \|x\|$ for any real number α

  (3) $\|x + y\| \leq \|x\| + \|y\|$

- Example: $x : [t_0, t_f] \longrightarrow \mathbb{R}$

$$\|x\| = \max_{t \in [t_0, t_f]} |x(t)|$$

- Example: $x : [t_0, t_f] \longrightarrow \mathbb{R}^n$

$$\|x\| = \int_{t_0}^{t_f} \left[x^T(t) x(t) \right]^{1/2} dt$$

. Example:

$$J(x(\cdot)) = \int_{t_0}^{t_f} \left[x^2(\tau) + 2x(\tau) \right] d\tau$$

$$\Delta J(x, \delta x) = J(x + \delta x) - J(x)$$

$$= \int_{t_0}^{t_f} \left\{ \left[x(\tau) + \delta x(\tau) \right]^2 + \cancel{2x(\tau)} + 2\delta x(\tau) \right\} d\tau + $$

$$- \int_{t_0}^{t_f} \left[x^2(\tau) + \cancel{2x(\tau)} \right] d\tau$$

$$= \int_{t_0}^{t_f} \left[\cancel{x^2(\tau)} + \delta^2 x(\tau) + 2\delta x(\tau) x(\tau) + 2\delta x(\tau) - \cancel{x^2(\tau)} \right] d\tau$$

$$= \underbrace{2 \int_{t_0}^{t_f} \left[\left(x(\tau) + 1 \right) \delta x(\tau) \right] d\tau}_{\text{- LINEAR IN } \delta x \quad \delta J(x, \delta x)} + \underbrace{\int_{t_0}^{t_f} \delta^2 x(\tau)\, d\tau}_{g(x, \delta x) \| \delta x \|}$$

⟶ it's very similar to the Taylor Expansion

. We want to minimize a function (with these examples); you have a general problem in which you have to optimize signal input of muscles to optimize the function in order to perform a given task.

this new concept of DERIVATIVE is the δJ for the functions.

- THEO (fundamental THEOREM of the CALCULUS of VARIATION)

if $x^* : [t_0, t_f] \longrightarrow \mathbb{R}^n$ is a EXTREMUM (i.e. maximum/minimum) then the VARIATION of J IN x^* is ZERO for ALL admissible[1] δx:

$$\delta J(x^*, \delta x) = 0 \qquad \forall \delta x$$

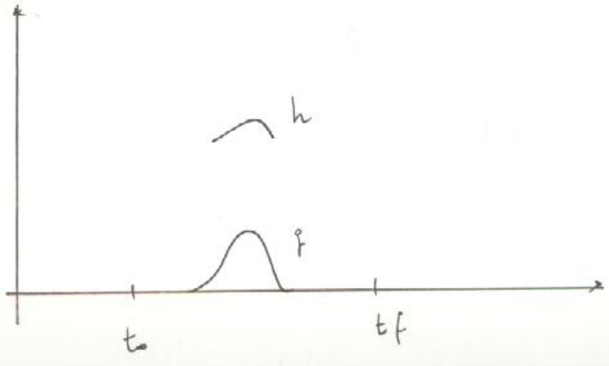
1 - important when we have constraints on the function

- Theorem: (Fundamental Lemma for the calculus of variation)

Given a function $h : [t_0, t_f] \longrightarrow \mathbb{R}$ if

$$\int_{t_0}^{t_f} h(\tau) g(\tau)\, d\tau = 0 \qquad \forall g : [t_0, t_f] \longrightarrow \mathbb{R}, \quad g \in C^0[t_0, t_f]$$

then: $h(t) = 0 \qquad \forall t \in [t_0, t_f]$

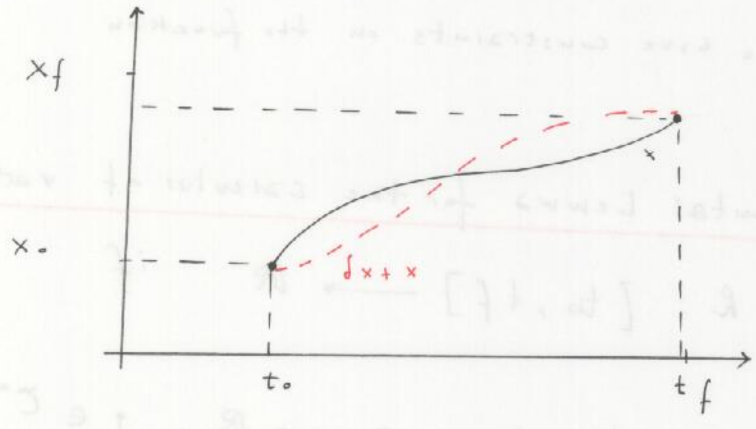


• Let's consider a particular problem:

$$\min_{x(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau \quad \text{s.t} \quad \begin{array}{l} x(t_0) = x_0 \\ x(t_f) = x_f \end{array}$$

$$\Delta J\left(x, \delta x\right) = \underbrace{\int_{t_0}^{t_f} L\left(x(\tau) + \delta x(\tau), \dot{x}(\tau) + \delta \dot{x}(\tau), \tau\right) d\tau}_{1} -$$

$$- \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau$$



I want to write the part 1 in a different way using the Taylor Expansion to:

$$L\left(x(\tau) + \delta x(\tau), \dot{x}(\tau) + \delta \dot{x}(\tau), \tau\right) = L\left(x(\tau), \dot{x}(\tau), \tau\right) +$$

$$+ \frac{\partial L}{\partial x}\left(x(\tau), \dot{x}(\tau), \tau\right) \delta x(\tau) + \frac{\partial L}{\partial \dot{x}}\left(x(\tau), \dot{x}(\tau), \tau\right) \delta \dot{x}(\tau)$$

$$+ \underbrace{o\left(\delta x(t), \delta \dot{x}(\tau)\right)}_{\substack{\text{Goes to zero as} \\ \delta x^2,\ \delta \dot{x}^2}}$$

ote that δx and $\delta \dot{x}$ are NOT INDEPENDENT! it's a technical detail, their relationship is:

$$\delta x(t) = \int_{t_0}^{t} \delta \dot{x}(\tau)\, d\tau + \delta x(t_0)$$

if we use the Expansion, we get:

$$\Delta J(x, \delta x) = \int_{t_0}^{t_f} \Big[\underbrace{\frac{\partial L}{\partial x}\big(x(\tau), \dot{x}(\tau), \tau\big)\delta x(\tau) + \frac{\partial L}{\partial \dot{x}}\big(x(\tau), \dot{x}(\tau), \tau\big)\delta \dot{x}(\tau)}_{\to\ \delta J(x, \delta x)}$$

$$+ \underbrace{o\big(\delta x(\tau), \delta \dot{x}(\tau)\big)}_{g(x, \delta x)\|\delta x\|} \Big] d\tau$$

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \Big[\underbrace{\frac{\partial L}{\partial x}\big(x(\tau), \dot{x}(\tau), \tau\big)\delta x(\tau)}_{(****)} + \overbrace{\underbrace{\frac{\partial L}{\partial \dot{x}}\big(x(\tau), \dot{x}(\tau), \tau\big)}_{f(\tau)} \underbrace{\delta \dot{x}(\tau)}_{\frac{d}{d\tau} g(\tau)}}^{(*)} \Big] d\tau$$

at this moment we have to force that $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial \dot{x}}$ are linked

$$\int_{t_0}^{t_f} \frac{d}{d\tau}\big(f(\tau)\, g(\tau)\big) d\tau = \underbrace{f(t)\, g(t)\Big|_{t_0}^{t_f}}_{(**)} = \underbrace{\int_{t_0}^{t_f} \frac{d}{d\tau} f(\tau)\, g(\tau)\, d\tau}_{(***)} + \underbrace{\int_{t_0}^{t_f} f(\tau) \frac{d}{d\tau} g(\tau)\, d\tau}_{(*)}$$

$\longrightarrow$ if we use this rule here (INTEGRATION by PARTS) i can REWRITE $\longrightarrow$ $g(t)f(t)\Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} f(\tau)\dot{g}(\tau)\,d\tau = \int_{t_0}^{t_f} \dot{f}(\tau) g(\tau)\, d\tau$

$$\int_{t_0}^{tf} \underbrace{\frac{\partial L}{\partial \dot{x}}\left(x(\tau), \dot{x}(\tau), \tau\right)}^{g(\tau)} \overbrace{\delta \dot{x}(\tau)}^{\dot{f}(\tau)} d\tau =$$

$$= \frac{\partial L}{\partial \dot{x}}\left(x(tf), \dot{x}(tf), tf\right)\delta x(tf) - \underbrace{\frac{\partial L}{\partial \dot{x}}\left(x(t_0), \dot{x}(t_0), t\right)\delta x(t_0)}_{(**)} -$$

$$- \underbrace{\int_{t_0}^{tf} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(x(\tau), \dot{x}(\tau), \tau\right)\delta x(\tau)\, d\tau}_{(***)}$$

$$\delta J(x, \delta x) = \frac{\partial L}{\partial \dot{x}}\left(x(tf), \dot{x}(tf), tf\right)\delta x(tf) - \frac{\partial L}{\partial \dot{x}}\left(x(t_0), \dot{x}(t_0), t_0\right)\delta x(t_0)$$

$$+ \int_{t_0}^{tf}\left[\underbrace{\frac{\partial L}{\partial x}\left(x(\tau), \dot{x}(\tau), \tau\right)}_{(****)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(x(\tau), \dot{x}(\tau), \tau\right)\right]\delta x(\tau)\, d\tau$$

$$\delta J(x^*, \delta x) = \frac{\partial L}{\partial \dot{x}}\left(x^*(tf), \dot{x}^*(tf), tf\right)\delta x(tf) - \frac{\partial L}{\partial \dot{x}}\left(x^*(t_0), \dot{x}^*(t_0) t_0\right)\delta x(t_0)$$

$$+ \int_{t_0}^{tf}\underbrace{\left[\frac{\partial L}{\partial x}\left(x^*(\tau), \dot{x}(\tau), \tau\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(x^*(\tau), \dot{x}^*(\tau), \tau\right)\right]}_{h}\underbrace{\delta x(\tau)}_{g}\, d\tau = 0$$

$$\forall \delta x$$

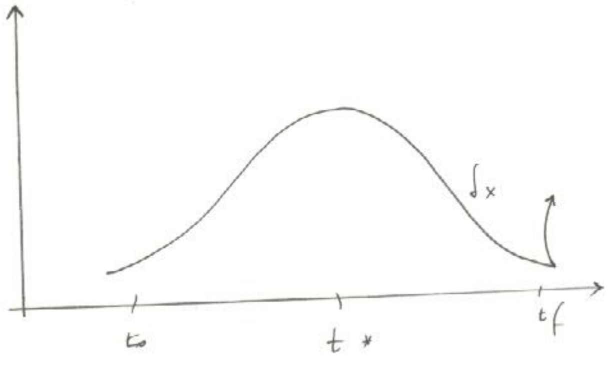
- <u>1° CASE</u>: $x(t_0) = x_0$, $x(tf) = xf \implies \delta x(t_0) = 0$, $\delta x(tf) = 0$
- IF WE REMEMBER $\int h g = 0 \implies h = 0$ and we call h e g the parts above, we obtain the EULER - LAGRANGE EQUATION:

$$\boxed{\frac{\partial L}{\partial x}\left(x^*(\tau), \dot{x}^*(\tau), \tau\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(x^*(\tau), \dot{x}^*(\tau), \tau\right) = 0}$$

the E-L Eq. is following from a minimization problem. and the function L is a characteristic of the system and depends on which system do we have, it's a general tool.

. <u>2nd CASE</u>: $x(t_0) = x_0$, $x(t_f)$ is free $\Rightarrow \delta x(t_0) = 0$, $\delta x(t_f)$ is arbitrary

a specific on the integral is not a specific on a specific point, if we have a function and we evaluate the integral we can evaluate the function in a specific point t^* without changing the integral.



I want to show that there is not link between $\delta x(t_0)$ e $\delta x(t_f)$ if we do a 'jump' in t_f the integral doesn't change.

$$\delta J(x^*, \delta x) = \frac{\partial L}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\delta x(t_f) + \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]\delta x(\tau)\, d\tau = 0 \qquad \forall\, \delta x, \delta x(t_f)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \qquad \forall\, t \in [t_0, t_f]$$

$$\frac{\partial L}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) \delta x(t_f) = 0 \quad \forall\, \delta x(t_f) \implies$$

$$\boxed{\frac{\partial L}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) = 0}$$