16 March 201
NORI

- BIOROBOTICS: Lesson 5

- Summary of what we did:

$$\min_{x(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau \quad \text{s.t.} \quad f\left(x(t), \dot{x}(t), t\right) = 0$$

$$f\left(x(t), t\right) = 0 \longrightarrow \text{(semplification of the minim. problem)}$$
: PARTICULAR CASE

If $x(t)$ satisfies the constraints, then $x(t) + \delta x(t)$ should satisfy the following in order to have

$$f\left(x(t) + \delta x(t), t\right) = 0:$$

TAYLOR EXP.:

$$\underbrace{f\left(x(t) + \delta x(t), t\right)}_{*} = \underbrace{f\left(x(t), t\right)}_{*} + \frac{\partial f}{\partial x}\left(x(t), t\right) \delta x(t) + o\left(\delta x(t)\right)$$

$$\underset{\text{FIRST ORDER APPROXIMATION}}{\Longrightarrow} \quad \frac{\partial f}{\partial x}\left(x(t), t\right) \delta x(t) = 0$$

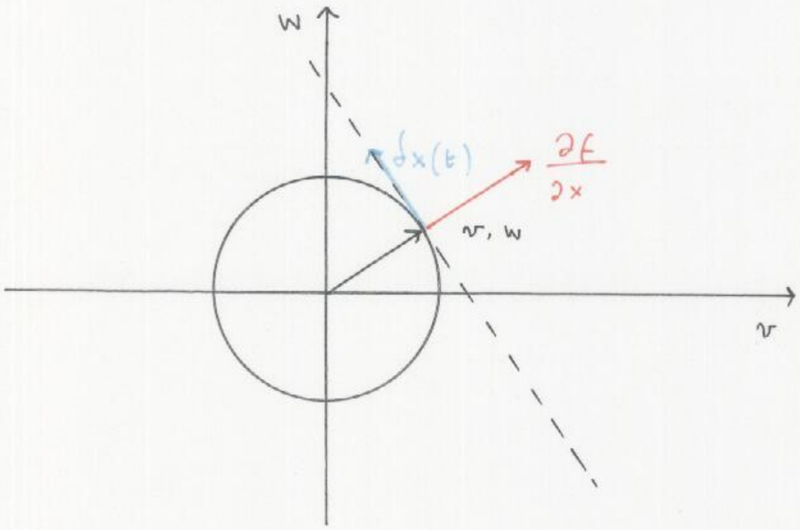
Instead of considering: $\min_{x(\cdot)} J(x)$

$$\Delta J(x, \delta x) = J(x + \delta x) - J(x)$$
$$= \delta J(x, \delta x) + g(x, \delta x) \|\delta x\|$$

$\longmapsto$ Like in this case, we neglect high order therms *

- Es: $x(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$, $\overbrace{v^2(t) + w^2(t) - R^2}^{f(x(t),\,t)} = 0$

$\quad \longrightarrow$ CONSTRAINT = CIRCLE



$$\frac{\partial f}{\partial x}\left(x(t), t\right) = \begin{bmatrix} \frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial w} \end{bmatrix} = \begin{bmatrix} 2v \\ 2w \end{bmatrix}$$

- THE EULER CONDITION:

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \underbrace{\left[\frac{\partial L}{\partial x}\left(x^*(\tau), \dot{x}^*(\tau), \tau\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(x^*(\tau), \dot{x}^*(\tau), \tau\right) \right]^T}_{(*)\ y^T} \underbrace{\delta x(t)}_{x}\, d\tau = 0$$

$$\forall\, \delta x(t) \quad \text{s.t.} \quad \underbrace{\frac{\partial f}{\partial x}\left(x^*(t), t\right)}_{A} \underbrace{\delta x(t)}_{x} = 0 \qquad \text{the CONSTRAINT}$$

$(*)$ has to be zero for every possible perturbation $\longrightarrow$ so:

. PROPERTY: If $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^n$

$$\text{if } y^T x = 0 \quad \forall x : A x = 0$$

$$\text{then } y \in \mathrm{Im}(A^T)$$

$$\exists\, p \in \mathbb{R}^m : A^T p = y$$

$$\Rightarrow \exists\, p(t) \text{ s.t. } \boxed{-\left[\frac{\partial f}{\partial x}\left(x^*(t), t\right)\right]^T p(t) = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}} \quad (*)$$

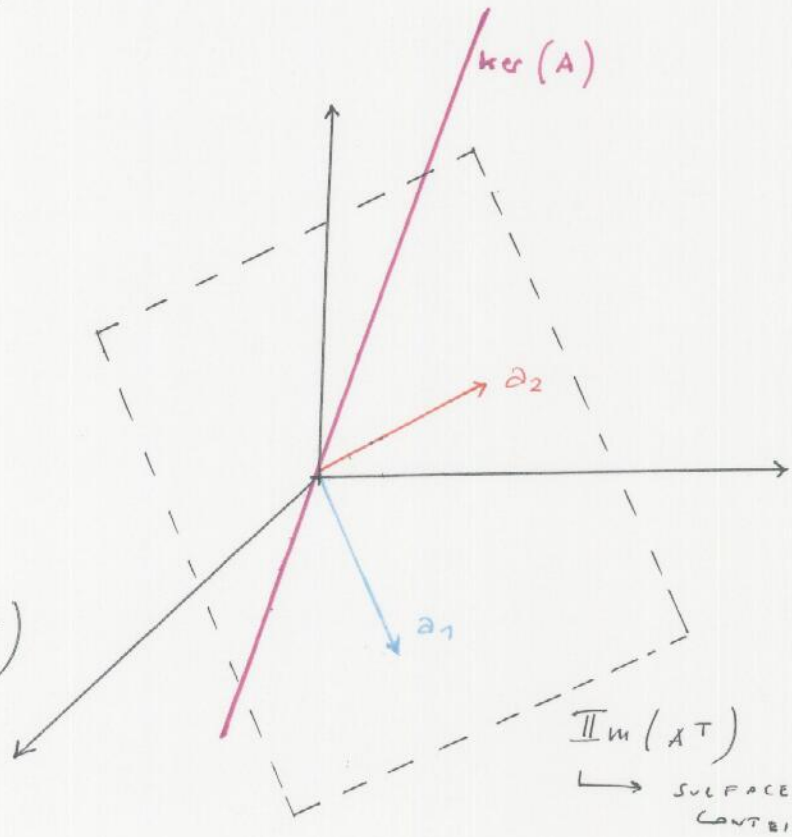
$\longmapsto$ LAGRANGE MULTIPLIERS

. Why do we need this:

$$A = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

$$\left[\ker(A)\right]^\perp = \mathrm{Im}(A^T)$$



$\hookrightarrow$ SURFACE PLANE CONTAINING a_i's.

$$La\left(x(t),\dot{x}(t),p(t),t\right)=L\left(x(t),\dot{x}(t),t\right)+p^T(t)\underbrace{f\left(x(t),t\right)}_{(**)}$$

$(*)$ is equivalent to $\frac{\partial La}{\partial x}\left(x(t)^*,\dot{x}^*(t),p^*(t),t\right)-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(x^*(t),\dot{x}^*(t),p^*(t),t\right)=0$

n.b. introduced the '$-$' in $(*)$, because if exists $p(t)$, exists also $-p(t)$

Everything it's the same if we sostitute $(**)$ with:

$$f\left(x(t),t\right)\longrightarrow f\left(x(t),\dot{x}(t),t\right)$$

- OPTIMAL CONTROL

$$La\left(x(t),\dot{x}(t),p(t),t\right)=L\left(x(t),\dot{x}(t),t\right)+p^T(t)\,f\left(x(t),\dot{x}(t),t\right)$$

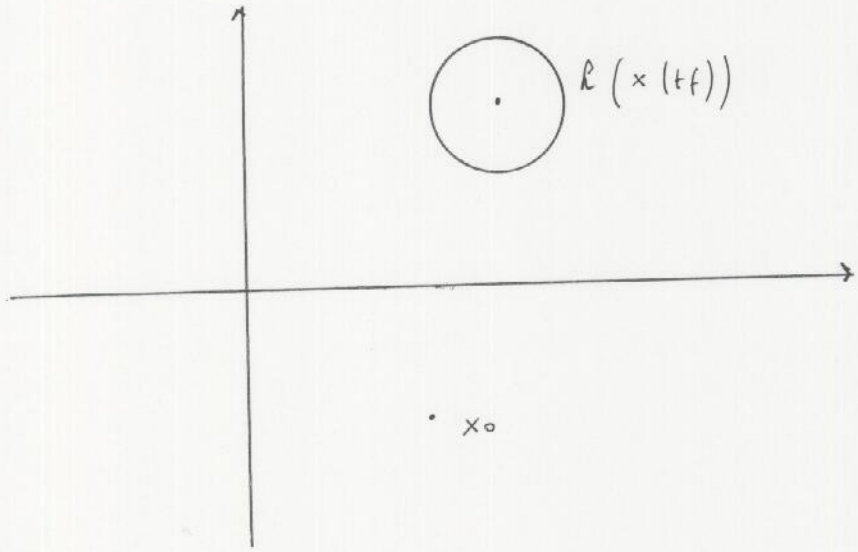
$$\min_{x(\cdot)}\int_{t_0}^{t_f}L\left(x(\tau),\dot{x}(\tau),\tau\right)d\tau$$

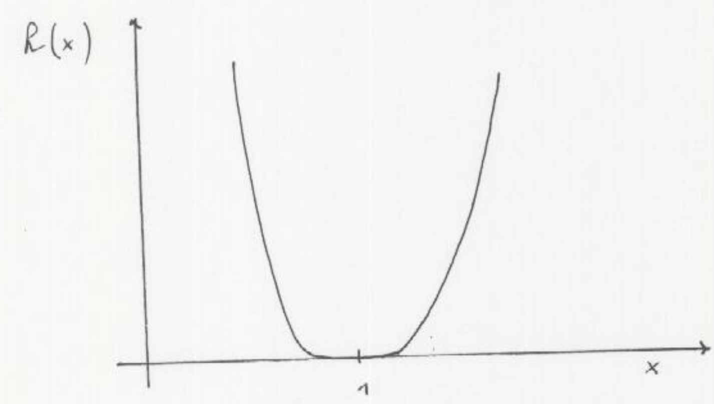
$$\text{s.t.}\quad f\left(x(t),\dot{x}(t),t\right)=0\quad,\quad x(t_0)=x_0$$

here the case is:

$$\min_u\int_{t_0}^{t_f}\overbrace{g\left(x(\tau),u(\tau),\tau\right)d\tau+h\left(x(t_f),t_f\right)}^{(***)}$$

$$\text{s.t.}\begin{cases}\dot{x}(t)=f\left(x(t),u(t),t\right)\\ x(t_0)=x_0\end{cases}\longrightarrow\begin{cases}\dot{x}=Ax+Bu\\ x(t_0)=x_0\end{cases}$$





the usual LINEAR STATE SYSTEM:

$$\longrightarrow \begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \end{cases}$$

LINEAR QUADRATIC REGULATOR:

(* * *)

$$g(x, u) = \frac{1}{2}\left(u^T R u + x^T Q x\right)$$

$$\min_{x(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau$$

$$\text{s.t.} \begin{cases} f\left(x(t), \dot{x}(t), t\right) = 0 \\ x(t_0) = x_0 \end{cases}$$

$\longmapsto$ to use this, instead of $x(t) \longrightarrow \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$

so:

$$\min_{u, x} \int_{t_0}^{t_f} g\left(x(t), u(t), \tau\right) d\tau + \underbrace{h\left(x(t_f), t_f\right)}$$

This is not an '$\int$'
and in min we have
something that is '$\int$' so:

$$h\left(x(t_f), t_f\right) = \int_{t_0}^{t_f} \frac{d}{d\tau} h\left(x(\tau), \tau\right) d\tau + h\left(x(t_0), t_0\right)$$

$$\min_{x, u} \int_{t_0}^{t_f} \underbrace{\left[g\left(x(\tau), u(\tau), \tau\right) + \frac{d}{d\tau} h\left(x(\tau), \tau\right) \right]}_{L} d\tau + \underbrace{\cancel{h\left(x(t_0), t_0\right)}}$$

it's a constant
and don't take
part in the minimizat.
will be multiplied for
δx_0 that will be ϕ

$$\text{s.t.} \begin{cases} \dot{x}(t) = f\left(x(t), u(t), t\right) \\ x(t_0) = x_0 \end{cases}$$

Now we have to compute:

$$L_a\left(x(t), \dot{x}(t), u(t), \dot{u}(t), p(t), t\right) = g\left(x(t), u(t), t\right) + \frac{d}{d\tau} h\left(x(t), t\right) + $$
$$+ p^T(t)\left[f\left(x(t), u(t), t\right) - \dot{x}(t)\right]$$

$$\frac{d}{dt}\left[h\left(x(t), t\right)\right] = \frac{\partial h}{\partial x}\left(x(t), t\right) \underbrace{\frac{dx}{dt}}_{\dot{x}(t)} + \frac{\partial h}{\partial t}\left(x(t), t\right)$$

$$L_a\left(x(t), \dot{x}(t), u(t), \dot{u}(t), p(t), t\right) = g\left(x(t), u(t), t\right) + \frac{\partial h}{\partial x}\left(x(t), t\right)\dot{x}(t) + $$
$$+ \frac{\partial h}{\partial t}\left(x(t), t\right) + p^T(t)\left[f\left(x(t), u(t), t\right) - \dot{x}(t)\right]$$

$$\frac{d}{dt}\left[h\left(x(t), y(t)\right)\right] = \frac{\partial h}{\partial x} \cdot \dot{x} + \frac{\partial h}{\partial y} \cdot \dot{y}$$

Now we compute the EULER LAGRANGE with $x \longrightarrow \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$

Let's do the computations;

$$\frac{\partial L_a}{\partial u}\left(x(t), \dot{x}(t), u(t), \dot{u}(t), t\right) = \frac{\partial g}{\partial u}\left(x(t), u(t), t\right) + \left[\frac{\partial f}{\partial u}\left(x(t), u(t), t\right)\right]^T \cdot p(t)$$

$$\frac{\partial L_a}{\partial \dot{u}} = 0$$

$$\frac{\partial L_a}{\partial x}\left(x(t), \dot{x}(t), u(t), \dot{u}(t), \quad t\right) = \frac{\partial g}{\partial x}\left(x(t), u(t), t\right) + \frac{\partial^2 h}{\partial x^2}\left(x(t), t\right)\dot{x}(t) +$$

$$+ \frac{\partial^2 h}{\partial x \partial t}\left(x(t), t\right) + \left[\frac{\partial f}{\partial x}\left(x(t), u(t), t\right)\right]^T p(t)$$

$$\frac{\partial L_a}{\partial \dot{x}}\left(x(t), \dot{x}(t), u(t), \dot{u}(t), t\right) = \frac{\partial h}{\partial x}\left(x(t), t\right) - p(t)$$

$$\frac{d}{dt}\frac{\partial L_a}{\partial \dot{x}}\left(x(t), \dot{x}(t), u(t), \dot{u}(t), t\right) = \frac{\partial^2 h}{\partial x^2}\left(x(t), t\right)\dot{x}(t) + \frac{\partial^2 h}{\partial x \partial t}\left(x(t), t\right) - \dot{p}(t)$$

• 1° CONDITION: (*)

$$\Rightarrow \frac{\partial L_a}{\partial x} - \frac{d}{dt}\frac{\partial L_a}{\partial \dot{x}} = \frac{\partial g}{\partial x}\left(x(t), u(t), t\right) + \left[\frac{\partial f}{\partial x}\left(x(t), u(t), t\right)\right]^T p(t) + \dot{p}(t) = 0$$

• 2° CONDITION: (**)

$$\frac{\partial L_a}{\partial u} - \frac{d}{dt}\frac{\partial L_a}{\partial \dot{u}} = \frac{\partial g}{\partial u}\left(x(t), u(t), t\right) + \left[\frac{\partial f}{\partial u}\left(x(t), u(t), t\right)\right]^T p(t) = 0$$

IF you define a FUNCTION that is HAMILTONIAN:

$$\mathcal{H}\left(x(t), u(t), p(t), t\right) = g\left(x(t), u(t), t\right) + p^T(t) f\left(x(t), u(t), t\right)$$

3 CONDITIONS ON THE HAMILTONIAN

$$\begin{cases} (*) & \dfrac{\partial \mathcal{H}}{\partial x}\left(x^*(t), u^*(t), p^*(t), t\right) = -\dot{p}^*(t) \\ (**) & \dfrac{\partial \mathcal{H}}{\partial u}\left(x^*(t), u^*(t), p^*(t), t\right) = 0 \\ (***) & \dfrac{\partial \mathcal{H}}{\partial p}\left(x^*(t), u^*(t), p^*(t), t\right) = f\left(x^*(t), u^*(t), t\right) = \dot{x}^*(t) \end{cases}$$

PONTRYAGIN'S MAXIMUM PRINCIPLE
[NECESSARY CONDITIONS for OPTIMALITY]

In case we left the FINAL STATE free and the time of execution, we have the:

- ADDITIONAL CONDITION FOLLOWING FROM $x(tf)$ is FREE:

$$\left[\frac{\partial h}{\partial x}\left(x^*(tf), tf\right) - p^*(tf)\right]^T \delta x f = 0 \quad \forall\ \delta_{xf}$$

$$\Rightarrow \quad p^*(tf) = \frac{\partial h}{\partial x}\left(x^*(tf), tf\right)$$

- ADDITIONAL CONDITION IF tf is FREE:

$$\mathcal{H}\left(x^*(tf), u^*(tf), p^*(tf), tf\right) + \frac{\partial h}{\partial t}\left(x^*(tf), tf\right) = 0$$

- CONSIDERATIONS:

I^{ST} CASE: $\min\limits_{u(\cdot)} \int_{t_0}^{tf} g\left(x(\tau), u(\tau), \tau\right) d\tau + h\left(x(tf), tf\right)$

$$\text{s.t.} \begin{cases} \dot{x}(t) = f\left(x(t), u(t), t\right) \\ x(t_0) = x_0, \quad xf \text{ is FREE}, \quad tf \text{ is FIXED} \end{cases} \qquad x \in \mathbb{R}^n \Longrightarrow p \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m$$

$$\begin{cases} \dot{p}^*(t) = -\dfrac{\partial \mathcal{H}\left(x^*(t), u^*(t), p^*(t), t\right)}{\partial x} & n\text{- Dynamical Equations} \\[2ex] \dot{x}^*(t) = \dfrac{\partial \mathcal{H}\left(x^*(t), u^*(t), p^*(t), t\right)}{\partial p} & n\text{- dynamical Equations} \\[2ex] \dfrac{\partial \mathcal{H}}{\partial u}\left(x^*(t), u^*(t), p^*(t), t\right) = 0 & m\text{- equations} \end{cases}$$

$\longleftrightarrow$ system of m equations and m unknowns, we compute:

$u^*\left(x(t), p(t), t\right) \longrightarrow$ if i sostitute this in the second eq. i'll have n- system, need BOUNDARY CONDITIONS:

1) $x^*(t_0) = X_0$

2) $\frac{\partial h}{\partial x}\left(x^*(tf), tf\right) - p^*(tf) = 0$

Since is a dynamical system with boundary conditions on t_0 and tf it's not trivial to solve it.

- Suppose we have the system:

$\dot{X} = AX$ if i have the CONDITION $X(0) = X_0 \longrightarrow$ I can solve it! n conditions and n dynamics EQ.

If I have:

$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = A \begin{bmatrix} x \\ p \end{bmatrix}$, $x(0) = x_0$ $p(T) = P_T$

the same n COND. and n EQ.

- IF I have the PROBLEM of the I^{ST} CASE but with

$x(tf) = xf \longrightarrow$ i have the initial state, the final state and we have an EXPLICIT analytical solution for systems.