- Biorobotics: Lesson N° 6

- Summary of Last Lesson:

- PONTRYAGIN'S PRINCIPLE:

$$\min_{u} \int_{t_0}^{t_f} g(x(\tau), u(\tau), \tau)\, d\tau + h(x(t_f), t_f)$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(t_0) = x_0 \end{cases} \qquad (\mathcal{P})$$

$$\mathcal{H}(x(t), u(t), p(t), t) = g(x(t), u(t), t) + p^T(t) f(x(t), u(t))$$

u^*, x^* is an extremum for $\mathcal{P}$ then

$$\begin{cases} \dot{x}^*(t) = \dfrac{\partial \mathcal{H}}{\partial p}(x^*(t), u^*(t), p^*(t), t) = A x^*(t) + B u^*(t) \\ -\dot{p}^*(t) = \dfrac{\partial \mathcal{H}}{\partial x}(x^*(t), u^*(t), p^*(t), t) = Q x^*(t) + A^T p^*(t) \\ 0 = \dfrac{\partial \mathcal{H}}{\partial u}(x^*(t), u^*(t), p^*(t), t) = R u^*(t) + B^T p^*(t) \end{cases}$$

(1) * Pag 3 The DYNAMIC of The SYSTEM 2 DYNAMIC EQ.

$\Longrightarrow$ ** Pag 3

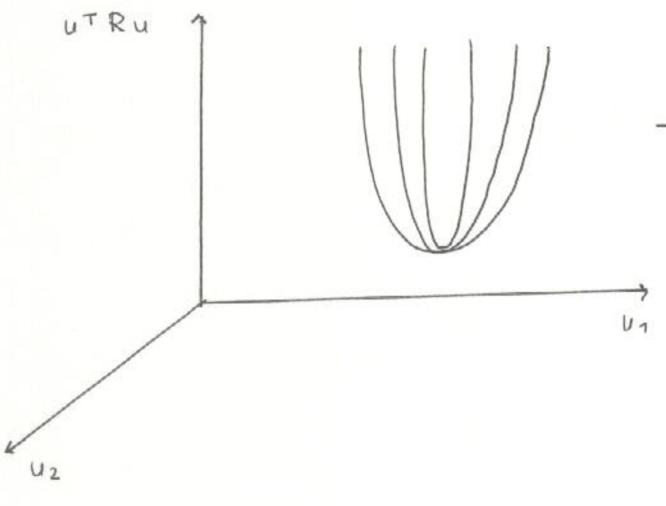
$$\underbrace{\left[\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)\right]}_{} \delta x_f + \left[\mathcal{H}(x^*(t_f), u^*(t_f), p^*(t_f), t_f) - \frac{\partial h}{\partial t}(x^*(t_f), t_f)\right] \delta t_f = 0$$

- LQR : LINEAR QUADRATIC REGULATOR

$$\min_{u} \int_{t_0}^{t_f} \underbrace{\left[\frac{1}{2} x^T(\tau) Q\, x(\tau) + \frac{1}{2} u^T(\tau) R\, u(t) \right]}_{\text{equivalent of: } g(x(\tau),\, u(\tau),\, \tau)} d\tau + \underbrace{\frac{1}{2} x^T(t_f)\, Q_f\, x(t_f)}_{\mathcal{L}(x(t_f),\, t_f)}$$

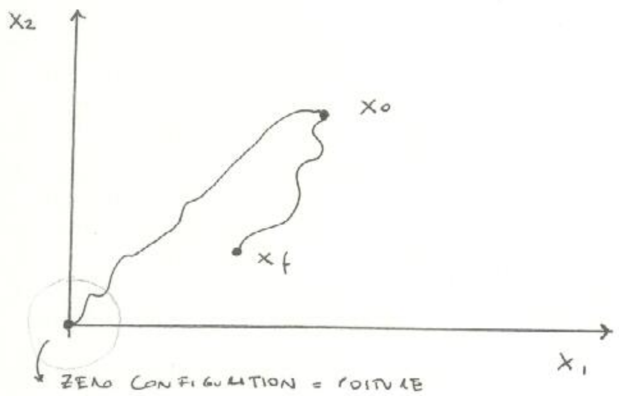
$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = \overbrace{A\, x(t) + B\, u(t)}^{f(x(t),\, u(t))} \\ x(t_0) = x_0 \end{cases}$$

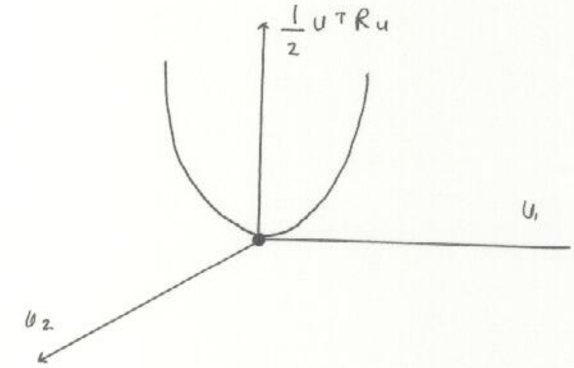
requirements: $Q = Q^T \geq 0, \quad Q_f = Q_f^T \geq 0, \quad R = R^T > 0 \quad \forall t$

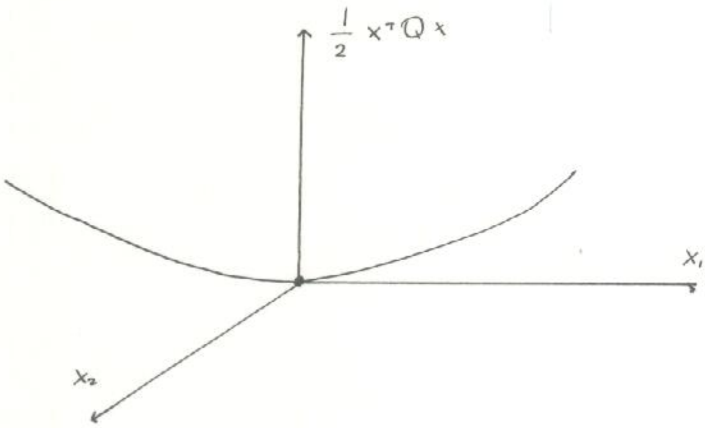


$\longrightarrow$ CONVEX

$$\mathcal{H}\left(x(t), u(t), p(t), t\right) = \underbrace{\frac{1}{2} x^T(t)\, Q\, x(t) + \frac{1}{2} u^T(t)\, R\, u(t)}_{g\left(x(t),\, u(t),\, t\right)} + p^T(t) \underbrace{\left[A\, x(t) + B\, u(t) \right]}_{f\left(x(t),\, u(t)\right)}$$







$\Rightarrow$ ** From pag 1:

$$u^*(t) = -R^{-1} B^T p^*(t)$$

*:

$$(1) \quad \begin{cases} \dot{x}^*(t) = A\, x^*(t) - B R^{-1} B^T p^*(t) \\ \dot{p}^*(t) = -Q x^*(t) - A^T p^*(t) \end{cases} \Longrightarrow \underbrace{\begin{bmatrix} \dot{x}^*(t) \\ \dot{p}^*(t) \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}}_{X(t)}$$

$$\dot{X}(t) = \mathcal{A} X(t), \qquad X(t_0) \longrightarrow X(t) = e^{\mathcal{A}(t-t_0)} X(t_0)$$

- CONDITIONS: $\triangle$ ∇

∇ $Q_f \cdot x^*(t_f) = p^*(t_f)$

$\triangle$ $x(t_0) = x_0$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

x_f is arbitrary, t_f is fixed

$\hookrightarrow$ so we have to solve a LINEAR SYSTEM with BOUNDARY CONDITIONS

- KALMAN: $\quad p^*(t) = K(t)\, x^*(t) \longrightarrow \dot{p}(t)^* = \dot{K}(t)\, x^*(t) + K(t)\, \dot{x}(t)$

Assuming that this is true,

$$\dot{x}(t) = \underbrace{Ax(t) - BR^{-1}B^T K(t)\, x(t)}$$

$$\dot{K}(t)\, x(t) + K(t)\, \dot{x}(t) = -Q\, x(t) - A^T K(t) \quad (t)$$

$$\dot{K}(t)\, x(t) + K(t) \left[A x(t) - BR^{-1}B^T K(t)\, x(t) \right] = -Q\, x(t) - A^T K(t)\, x(t)$$

$$\left[\dot{K}(t) + K(t) A - K(t) BR^{-1}B^T K(t) + A^T K(t) + Q \right] x(t) = 0$$

if we satisfy:

$$\begin{cases} \dot{K}(t) = -K(t) A - A^T K(t) + K(t) BR^{-1}B^T K(t) - Q & \boxed{\text{KALMAN EQUATION}} \\ K(t_f) = Q_f \quad \text{comes from } \nabla \text{ and from } p(t) = K(t)\, x(t) \end{cases}$$

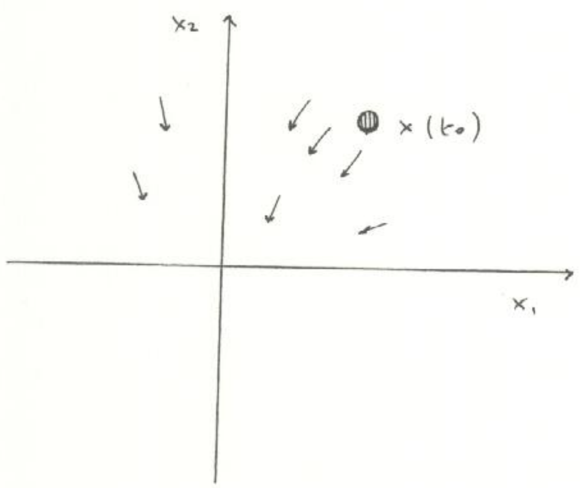
$$\begin{cases} \dot{x}^* = \left[A - B R^{-1} B^T k(t) \right] x(t) \\ x(t_0) = x_0 \end{cases}$$

$$u^*(t) = - R^{-1} B^T k(t)\, x(t)$$

. Suppose that I have a Dynamic System:

$$\begin{cases} \dot{x}(t) = f\left(x(t) \right) & x \in \mathbb{R}^n \\ x(t_0) = x_0 \end{cases} \qquad (1)$$

Example: a mass in a force field f



to solve the (1), simple cases:

$$\begin{cases} \dot{x} = -x \\ x(t_0) = x_0 \end{cases} \longrightarrow X(t) = e^{-t} x_0$$

$$\begin{cases} \dot{x} = -x^2 \\ x(t_0) = x_0 \end{cases} \longrightarrow \text{For solving this anatically is:}$$

$$\dot{x} = \frac{d}{dt} x(t) = \lim_{\delta \to 0} \frac{x(t+\delta) - x(t)}{\delta} \cong \frac{x(t+\delta) - x(t)}{\delta} \quad \text{for suff. small } \delta$$

$$\dot{x} = f(x(t)) \longrightarrow \frac{\overbrace{x(t + k\delta)}^{x_{k+1}} - \overbrace{x(t + (k-1)\delta)}^{x_k}}{\delta} = f\left(\overbrace{x(t + (k-1)\delta)}^{x_k}\right)$$

$$\boxed{x_{k+1} = x_k + \delta f(x_k)}$$