11 MARZO 2011
NOKI

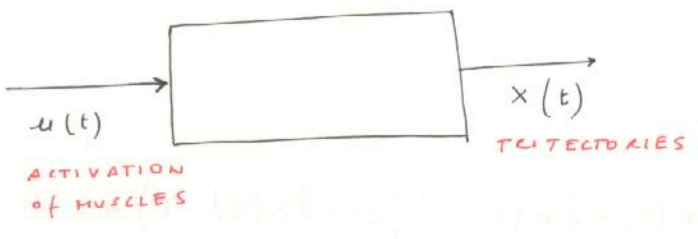
- BIOROBOTICS : Lesson 4

- $$\min_{x(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau)\, \tau\right) d\tau \qquad x(t_0) = x_0 \ , \ x(t_f) = x_f$$

- Problem of SOLVING an OPTIMAL CONTROL PROBLEM:

$$\min_{u(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), u(\tau), \tau\right) d\tau \quad \text{s.t.}$$

$$\text{s.t.} \quad \dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right) u(t)$$ Dynamical system



$$\dot{x}(t) = A\, x(t) + B\, u(t)$$

- $$\min_{x(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau \quad \text{s.t.}$$

$$f\left(x(t), \dot{x}(t), t\right) = 0 \quad \forall\, t \in [t_0, t_f]$$

$$\min_{x(\cdot)} \int_{t_0}^{tf} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau \qquad x(t_0) = x_0 , \quad x(tf) = xf$$

$$\delta J\left(x, \delta x\right) = \frac{\partial L}{\partial \dot{x}}\left(x(tf), \dot{x}(tf), tf\right) \delta x(tf) - \frac{\partial L}{\partial \dot{x}}\left(x(t_0), \dot{x}(t_0), t_0\right) \delta x(t_0) +$$

$$+ \int_{t_0}^{tf} \left[\frac{\partial L}{\partial x}\left(x(\tau), \dot{x}(\tau), \tau\right) - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}}\left(x(\tau), \dot{x}(\tau), \tau\right) \right] \delta x(\tau)\, d\tau$$

$$\delta J\left(x^*, \delta x\right) = 0 \qquad \forall \text{ ADMISSIBLE } \delta x$$

· 3° CASE: $x(t_0) = x_0$, xf is ARBITRARY, tf is FREE

So the DURATION of the MOVEMENT is ARBITRARY

DISCRETE INCREMENT:

$$\Delta J\left(x, \delta x, \delta tf\right) = \int_{t_0}^{tf + \delta tf} L\left(x(\tau) + \delta x(\tau), \dot{x}(\tau) + \delta \dot{x}(\tau), \tau\right) d\tau +$$

$$- \int_{t_0}^{tf} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau =$$

$$\boxed{\int_{t_0}^{tf + \delta tf} = \int_{t_0}^{tf} + \int_{tf}^{tf + \delta tf}}$$

$$= \frac{\partial L}{\partial \dot{x}}\left(x(tf), \dot{x}(tf), tf\right) \delta x(tf) - \frac{\partial L}{\partial \dot{x}}\left(x(t_0), \dot{x}(t_0), t_0\right) \delta x(t_0) +$$

$$+ \int_{t_0}^{tf} \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right] \delta x(\tau)\, d\tau + \underbrace{\int_{tf}^{tf + \delta tf} L\left(x(\tau) + \delta x(\tau), \dot{x}(\tau) + \delta \dot{x}(\tau), \tau\right) d\tau}_{\mathcal{L}(\delta tf)}$$

We have to do the Taylor Expansion and then Integrating by parts

$$\boxed{F(x) = \int_a^{a+x} f(t)\, dt \longrightarrow \left.\frac{dF}{dx}\right|_{x=0} = f(a)}$$

$$\mathcal{L}(\delta tf) = \int_{tf}^{tf+\delta tf} L\left(x(\tau)+\delta x(\tau), \dot{x}(\tau)+\delta\dot{x}(\tau), \tau\right) d\tau = \text{[TAYLOR EXPANSION AROUND } \delta tf = 0\text{]}$$

$$\mathcal{L}(0) + \left.\frac{\partial \mathcal{L}}{\partial \delta tf}\right|_{\delta tf = 0} \delta tf + o(\delta tf) =$$

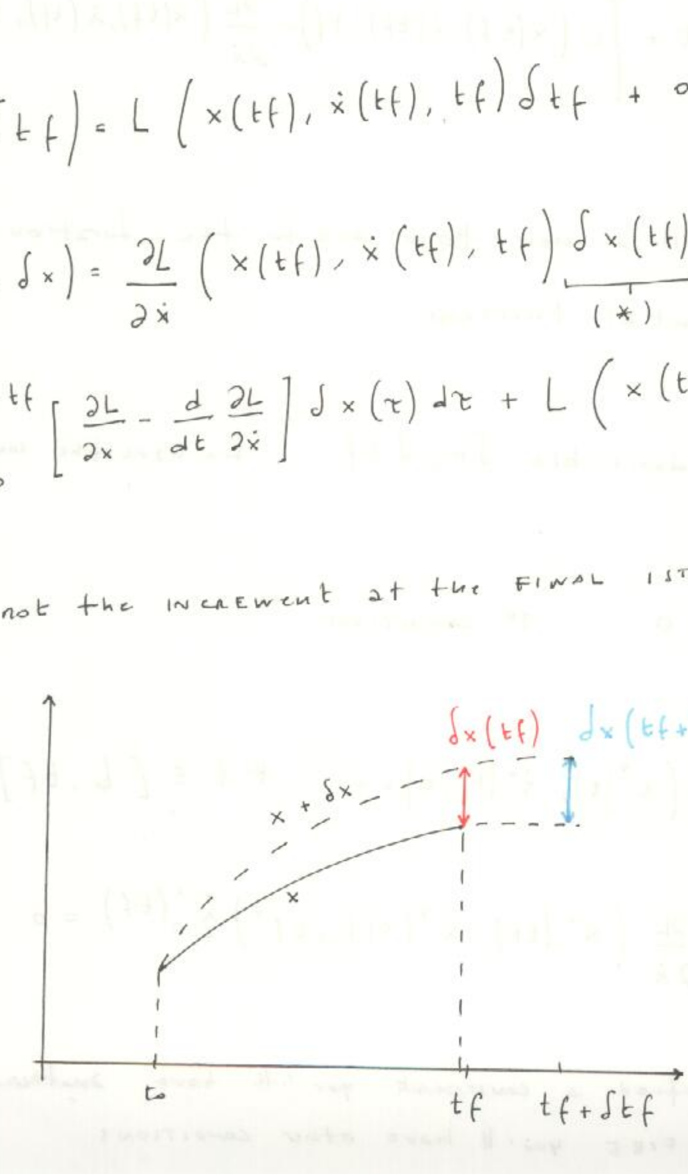
$$0 + L\left(x(tf)+\delta x(tf), \dot{x}(tf)+\delta\dot{x}(tf), tf\right) \delta tf + o(\delta tf)$$

$$\mathcal{L}(\delta tf) = L\left(x(tf), \dot{x}(tf), tf\right)\delta tf + o\left(\delta x(tf), \delta\dot{x}(tf), \delta tf\right)$$

$$\delta J(x, \delta x) = \frac{\partial L}{\partial \dot{x}}\left(x(tf), \dot{x}(tf), tf\right)\underbrace{\delta x(tf)}_{(*)} - \frac{\partial L}{\partial \dot{x}}\left(x(to), \dot{x}(to), to\right)\delta x(to) +$$

$$+ \int_{to}^{tf}\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]\delta x(\tau)\, d\tau + L\left(x(tf), \dot{x}(tf), tf\right)\delta_f$$

(*) is not the INCREMENT at the FINAL ISTANT, because it would be



$$\delta x f = \delta x (tf + \delta tf) = \underbrace{\int_{tf}^{tf+\delta tf} \delta \dot{x}(\tau)\, d\tau}_{\dot{x}(tf)\,\delta tf + o(\delta tf)} + \delta x (tf)$$

$$= \delta \dot{x}(tf) + o(\delta tf) + \delta x (tf)$$

$$\delta x(tf) = \delta x f - \dot{x}(tf)\delta tf + o(\delta tf)$$

subtituting this :

$$\delta J(x, \delta x) = \frac{\partial L}{\partial \dot{x}}\Big(x(tf), \dot{x}(tf), tf\Big)\delta x f - \frac{\partial L}{\partial \dot{x}}\Big(x(t_0), \dot{x}(t_0), t_0\Big)\delta x(t_0) +$$

$$+ \int_{t_0}^{tf}\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]\delta x(\tau)\, d\tau + \left[L\big(x(tf), \dot{x}(tf), tf\big) - \frac{\partial L}{\partial \dot{x}}\big(x(tf), \dot{x}(tf), tf\big)\,\dot{x}(tf)\right]\delta tf$$

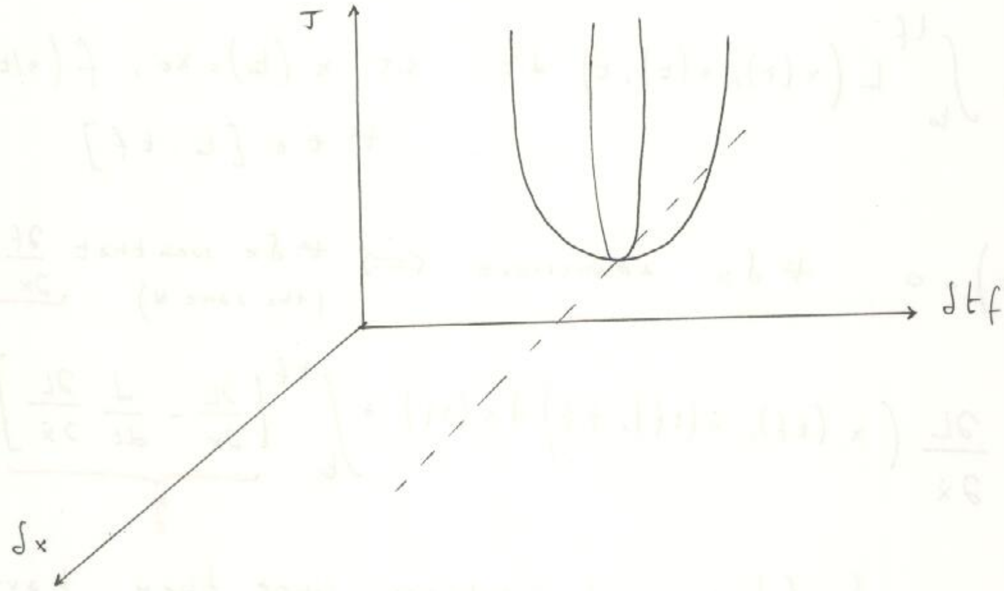
it's like before but there is a new term due to the duration of the movement. Using the fondamental theorem :

$\delta J(x^*; \delta x, \delta tf) = 0 \quad \forall$ admissible $\delta x, \delta tf$, we have the CONDITIONS:

- (1) $\frac{\partial L}{\partial \dot{x}}\Big(x^*(tf), \dot{x}^*(tf), tf\Big) = 0$ 1° CONDITION
- (2) $\frac{\partial L}{\partial x}\Big(x^*(t), \dot{x}^*(t), t\Big) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\Big(x^*(t), \dot{x}^*(t), t\Big) = 0 \quad \forall t \in [t_0, tf]$
- (3) $L\Big(x^*(tf), \dot{x}^*(tf), tf\Big) - \frac{\partial L}{\partial \dot{x}}\Big(x^*(tf), \dot{x}^*(tf), tf\Big)\dot{x}^*(tf) = 0$

$\longrightarrow$ if you dont specified a constraint you'll have another
if you have more FREE you'll have other CONDITIONS

Visualization of the problem:



⟶ it is the Extension of the CONCEPT of DERIVATE.
When you want the final position, the initial position, the trajectory and the cost function

- In the case of $x(\cdot)$ vector in $\mathbb{R}^n$, an extreme point x^* have to satisfy the conditions:

$$x(\cdot) : [t_0, t_f] \longrightarrow \mathbb{R}^n$$

$$\frac{\partial L}{\partial x}\left(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\left(x^*(t), \dot{x}^*(t), t\right) = 0$$

$$\left[\frac{\partial L}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]^T \delta x_f + \left\{ L\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) + \right.$$

$$\left. - \left[\frac{\partial L}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]^T \dot{x}^*(t_f) \right\} \delta t_f = 0 \quad \forall \delta t_f, \delta x_f$$

- CONSTRAINED FUNCTIONAL OPTIMIZATION:

$$x^*(\cdot) = \arg\min_{x(\cdot)} \int_{t_0}^{t_f} L\left(x(\tau), \dot{x}(\tau), \tau\right) d\tau \quad \text{s.t. } x(t_0) = x_0, \; f\left(x(t), t\right) = 0 \quad \forall\, t \in [t_0, t_f]$$

$$\delta J\left(x^*, \delta x\right) = 0 \quad \forall\, \delta x \text{ ADMISSIBLE} \iff \forall\, \delta x \text{ such that } \underbrace{\frac{\partial f}{\partial x}\left(x(t), t\right)}_{A} \underbrace{\delta x(t)}_{x} =$$

(the same $*$)

$$\delta J\left(x, \delta x\right) = \frac{\partial L}{\partial \dot{x}}\left(x(t_f), \dot{x}(t_f), t_f\right) \delta x(t_f) + \int_{t_0}^{t_f} \underbrace{\left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]}_{y} \underbrace{\delta x(\tau)}_{x} d\tau$$

but the variations $\delta x(\cdot)$ are not arbitrary since they have to preserve the constraint $f\left(x(t), t\right) = 0 \quad \forall\, t \in [t_0, t_f]$

$$f\left(x(t), t\right) = 0 \implies f\left(x(t) + \delta x(t), t\right) = 0 = \text{[TAYLOR EXPANS.]}$$

$$= \cancel{f\left(x(t), t\right)} + \frac{\partial f}{\partial x}\left(x(t), t\right) \delta x(t) + o\left(\delta x(t)\right)$$

$\downarrow \ddot{0}$

function of the value of the function in this istant time

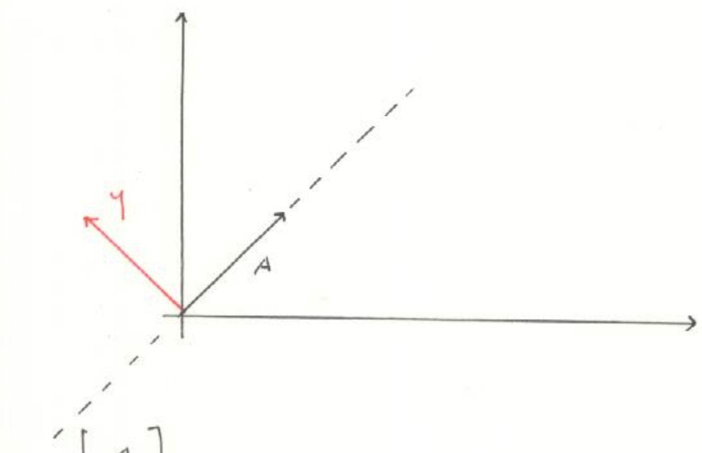
$$\overset{(*)}{\implies} \frac{\partial f}{\partial x}\left(x(t), t\right) \delta x(t) = 0 \quad \forall\, t \in [t_0, t_f]$$

- FACT: Given a matrix $A \in \mathbb{R}^{n \times m}$ and a vector $y \in \mathbb{R}^n$ if:

$$y^T x = 0 \quad \forall\, x : A x = 0$$

Then $y \in \mathrm{Im}\left(A^T\right)$ or equivalently

$$\exists\, p \in \mathbb{R}^n \text{ such that } A^T p = y$$



$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Dim: $\left[\ker (A) \right]^{\perp} = \operatorname{Im} \left(A^T \right)$

$$\ker(A) = \left\{ x \in \mathbb{R}^m : Ax = 0 \right\} \qquad \operatorname{Im}(A) = \left\{ w : \exists v \text{ t.c. } Av = w \right\}$$

$$y^T x = 0 \qquad \forall x : Ax = 0$$

$$\ldots \quad y^T x = 0 \qquad \forall x \in \ker(A)$$

$$y \in [\ker(A)]^{\perp} = \operatorname{Im}(A^T)$$

$\Longrightarrow \ \exists \, p(t) \quad t \in [t_0, t_f] \quad$ s.t.

$$\boxed{\left[\frac{\partial f}{\partial x} \left(x(t), t \right) \right]^T p(t) = \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}} \qquad (*)$$

$p(t) \longrightarrow$ LAGRANGE MULTIPLIER (UNKNOWN FUNCTION TO BE determined)

$$L_\alpha \left(x(t), \dot{x}(t), p(t), t \right) = L \left(x(t), \dot{x}(t), t \right) + p^T(t) f \left(x(t), t \right)$$

$$(*) \quad \frac{\partial L_\alpha}{\partial x} \left(x^*(t), \dot{x}^*(t), p(t), t \right) - \frac{d}{dt} \frac{\partial L_\alpha}{\partial \dot{x}} \left(\dot{x}(t), \ddot{x}(t), p(t), t \right) = 0$$