Blorobotics: Lesson 2

- Desert Morillo → MIT: With the "Baggio & Ferraro" or "MIT-HAMs", minimal trajectory of a point to point movement

\[ y \]

\[ (x_0, y_0) \quad (x_f, y_f) \]

- Linear trajectory

\[ \| \vec{v} \| \quad \| \vec{a} \| \]

\[ t \quad t_f \]

- \( x(t), y(t) \) is such that

\[ \min \int_{t_0}^{t_f} \left[ x''(t)^2 + y''(t)^2 \right] dt \quad \text{s.t.} \quad \text{MIN - JERK} \]
\[
\begin{align*}
\begin{cases}
  x(t_0) = x_0, & y(t_0) = y_0 \\
  x(t_1) = x_f, & y(t_1) = y_f
\end{cases}
\end{align*}
\]

Today we solve the min of J, we are optimizing over all possible trajectories. (It is a minimizing feature, a way to make it smooth and more conservative).

See 2 papers on the website: "HORASSO"

"HOOGAN"

"Old" racers

The tool we use is: \textbf{CALCULUS OF VARIATIONS}

If you say that

\[
x^*(t) = \arg\min_{x(t)} \int_{t_0}^{t_f} L(x(t), \dot{x}(t), t) \, dt
\]

is the traj. that minimizes

\[
\begin{align*}
\begin{cases}
  x(t_0) = x_0 \\
  x(t_1) = x_f
\end{cases}
\end{align*}
\]

\[x : [t_0, t_f] \to \mathbb{R}^n\]

If \(x^* : [t_0, t_f] \to \mathbb{R}^n\) is a maximum or a minimum (i.e. extremum) then:
\[
\begin{aligned}
\text{then: } & \frac{2L}{\partial x} (x^*(t), x^*(t), t) - \frac{1}{4t} \frac{\partial L}{\partial x} (x(t), x^*(t), t) = 0 \\
& \forall t \in [t_0, t_f]
\end{aligned}
\]

\[
\begin{aligned}
\begin{align*}
X^*(t_0) &= x_0 \\
X^*(t_f) &= x_f
\end{align*}
\end{aligned}
\]

Some applications to have in mind of how strong it is:

Example: If I want to go from a point to another, with a minimum length

\[
X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
\]

\[
\begin{aligned}
\text{arg min} \quad & \int_{t_0}^{t_f} \left[ \dot{x}^2(t) + \dot{y}^2(t) \right]^{1/2} \, dt \\
\text{subject to} \quad & x(t_0) = x_0 \\
& x(t_f) = x_f \\
& y(t_0) = y_0 \\
& y(t_f) = y_f
\end{aligned}
\]

\[
L \left( (x(t), \dot{y}(t), y(t), \dot{x}^2(t), t) \right)
\]

more precisely:

\[
L \left( (x(t), \dot{y}(t), \dot{x}(t), \dot{y}(t), t) \right)
\]

\[
= \left[ \dot{x}^2(t) + \dot{y}^2(t) \right]^{1/2}
\]
\[
\begin{align*}
\frac{\partial L}{\partial x} &= \begin{bmatrix}
\frac{\partial L}{\partial x} \\
\frac{\partial L}{\partial y}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \\
\frac{\partial L}{\partial x} &= \begin{bmatrix}
\frac{1}{2} \left[ x^2(t) + y^2(t) \right]^{1/2} \\
\frac{1}{2} \left[ x^2(t) + y^2(t) \right]^{1/2}
\end{bmatrix} \\
\frac{1}{2} \left( -\frac{1}{2} \right) \cdot \frac{1}{\left[ x^2(t) + y^2(t) \right]^{3/2}} \left( 2x\dddot{x} + 2y\dddot{y} \right) &= \frac{2\dddot{x}^2}{\left[ x^2(t) + y^2(t) \right]^{3/2}} \\
\frac{1}{dt} \frac{\partial L}{\partial x} &= \begin{bmatrix}
\frac{\dot{y}}{\left[ x^2 + y^2 \right]^{3/2}} \\
\frac{\dot{x}\dot{y}}{\left[ x^2 + y^2 \right]^{3/2}} - \frac{\dot{x}}{\left[ x^2 + y^2 \right]^{1/2}} \\
\frac{\dot{x}}{\left[ x^2 + y^2 \right]^{3/2}} - \frac{\dot{y}}{\left[ x^2 + y^2 \right]^{1/2}}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix} \\
\dot{y} \left( \dot{y} \dot{x} - \dot{x} \dot{y} \right) &= 0
\end{align*}
\]

Excluding the case \( \dot{x} = \dot{y} = 0 \) (still point that doesn't extend the curve) \( \rightarrow \) will be:
Assume that:

\[ y'' x' - x' y'' = 0 = \frac{d}{dt} \left( \frac{\dot{x}}{\dot{y}} \right) \cdot \dot{y}^2 = \frac{d}{dt} \left( \frac{\dot{x}}{\dot{y}} \right) \cdot \dot{y}^2 = \]

\[ = \frac{d}{dt} \left( \frac{\dot{x}}{\dot{y}} \right) \cdot \dot{y}^2 = 0 \]

\[ \Rightarrow \frac{d}{dt} \left( \frac{\dot{x}}{\dot{y}} \right) \cdot \dot{y}^2 = 0 \]

\[ \therefore \text{The wave that connects the two points is a line!} \]

**Least Action Principle**

In dynamical systems, when moving from a point to another, minimize the action.

Given a physical system we can associate to the system a mathematical quantity called the "action" which is an attribute of the system (it is a property of the dynamical system).

The "action" is a functional, i.e., an operator which takes as input trajectories of the system \( q(t) \) \( t \in [t_0, t_f] \) and associates real numbers to the trajectory.
A way to represent the action is \( S \) consists in integrating a function \( L (q(t), q'(t), t) \) of the system generalized coordinates \( q(t) \) and generalized velocities \( q'(t) \):

\[
S [q(t)] = \int_{t_0}^{t_f} L (q(t), q'(t), t) \, dt
\]

The operator \( S \) defines you a scalar, when the action of a given dynamical system can be described in this way, the function \( L(\cdot, \cdot, \cdot) \) is called the system Lagrangian.

**Theorem (Hamilton's Principle):** The true evolution of a mechanical system described by the generalized coordinates \( q(t) \in \mathbb{R}^n \) between two points \( q_0 \) and \( q_f \) is a stationary point of the action functional:

\[
S [q(t)] = \int_{t_0}^{t_f} L (q(t), q'(t), t) \, dt
\]

If you have a real system, to go to \( q_0 \) to \( q_f \) the function moves from a minimum to a max.

\( S \) has to satisfy:

\[
\begin{cases} 
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial q'} = 0 \\
q(t_0) = q_0, \quad q(t_f) = q_f
\end{cases}
\]

A point where local modification of the trajectory \( q(t) \) corresponds to zero variation.
Example: if the system is a kinematic chain with $n$ degree of freedom, then,

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} - V(q)$$

the way to compute the dynamic of the system you get to usual:

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + g(q) = 0$$

Euler-Lagrange equation it's a way to compute the dynamic equation. Imposing the necessary conditions for a stationary point of the associated action:

$$\frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}$$

we exactly get the system dynamics.

**Noether Theorem** --- if you look at the system, the Lagrange, and if you consider the Lagrange moved by the $q$ the system doesn't change (if you have symmetry in the function)

Look at it!

Example: consider the problem of throwing a wire and describing the dynamic

$$q(t) = \begin{bmatrix} x(t) \\ \dot{y}(t) \end{bmatrix}$$

$$L = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) - m y \dot{y}$$
If you apply Newton's Law we already know that the system dynamics are regulated by:

$$\begin{align*}
\dot{v} &= 0 \\
\dot{y} &= -mg
\end{align*}$$

which gives the system evolution from:

$$\begin{align*}
x(0) &= x_0, & y(0) &= y_0 \\
\dot{x}(0) &= \dot{x}_0, & \dot{y}(0) &= \dot{y}_0
\end{align*}$$

i.e.

From the least action principle, the action of the trajectory $q$ is:

$$S[q(t)] = \int_{t_0}^{t_f} \left[ \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - mg y \right] dt$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = -\dot{x} = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = -mg \dot{y} = 0$$

Using this theorem you can write:

$$\begin{align*}
\dot{v} &= 0 \\
\dot{x}(0) &= x_0, & y(0) &= y_0, & x(t) &= x, & y(t) &= y
\end{align*}$$

Therefore dynamics of the system are the same but boundary conditions are different but in both cases a unique solution of the dynamic equations can be determined.
Generalization of Calculus of Variations:

\[ x^* (t) = \arg \min_{x(1)} \int_{t_0}^{t_f} L \left( x(t), \dot{x}(t), \ldots, x^{(n)}(t); \tau \right) dt \]

s.t. \[ x(t_0) = x_0, \quad x(t_f) = x_f \quad \ldots \quad x^{(m-1)}(t_0) = x^{(m-1)}_0, \quad x^{(m-1)}(t_f) = x^{(m-1)}_f \]

\[ \frac{\delta L}{\delta x} - \frac{d}{dt} \frac{\delta L}{\delta \dot{x}} + \ldots + (-1)^{m-1} \frac{d^{m}}{dt^{m}} \frac{\delta L}{\delta x^{(m)}} = 0 \]

\[ x(t_0) = x_0, \quad \ldots \quad x^{(m-1)}(t_0) = x^{(m-1)}_0 \]

Example of the minimal jerk trajectory:

\[ \min_{x(1)} \int_{t_0}^{t_f} \left[ x^{(1)}(t)^2 + \left( x^{(2)}(t) + y^{(1)}(t) \right)^2 \right] dt \]

We need to min this function, with boundary conditions:

\[ x(t_0) = x_0 \quad x(t_f) = x_f \]

\[ y(t_0) = y_0 \quad y(t_f) = y_f \]
The integral is a linear function and can be decomposed into two parts:

\[
\min_x \int x^{(2)} + \min_y \int y^{(2)}
\]

I can basically solve one problem because the other is the same:

\[
\min_x \frac{1}{2} \int_{x(1)}^{T} \left[ x^{(2)} \right]^2 \, \text{d}\tau
\]

We assume that the others initial conditions are \( \dot{\mathbf{x}}(0) = 0 \), \( \dot{\mathbf{x}}(T) = 0 \)

\[
\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(T) = \mathbf{x}_T
\]

The Lagrangian:

\[
L\left( x(t), \dot{x}(t), x^{(2)}(t), x^{(3)}(t), t \right) = \left[ x^{(3)}(t) \right]^2
\]

\[
\frac{\partial L}{\partial x} = 0
\]

\[
\frac{\partial L}{\partial \dot{x}} = 0
\]

\[
\frac{\partial L}{\partial x^{(2)}} = 0
\]
\[
\frac{dL}{dx^{(1)}} = 2 \cdot x^{(1)}(t) - \frac{1}{x^{(1)}}
\]

\[
(-1)^3 \frac{d^3 L}{dt^3} \frac{dL}{dx^{(1)}} = (-1) \cdot x^{(4)}(t) = 0 \quad \Rightarrow \quad x(t) = a_0 + a_1 t + \ldots + a_5 t^5
\]

**Solution:** Polynomial 5th order

**Linear Differential Equation**

\[
x(0) = x_0 = a_0
\]
\[
x'(0) = 2a_1 = 0
\]
\[
x''(0) = 2a_2 = 0
\]

\[
a_0 = x_0, \quad a_1 = a_2 = 0
\]

\[
x(T) = x_0 + 2a_1 T^2 + 2a_2 T^4 + 2a_3 T^5 = x_T
\]
\[
x'(T) = 3a_2 T^2 + 4a_4 T^3 + 8a_5 T^4 = 0
\]
\[
x''(T) = 6a_3 T + 12a_4 T^2 + 20a_5 T^3 = 0
\]

\[
3a_3 T^2 = -4a_4 T^2 - 5a_5 T^4 \quad \Rightarrow \quad 3a_3 T^2 = -\frac{4}{3} a_4 T^2 - \frac{5}{3} a_5 T^4
\]

\[
-8a_4 T^2 - 10a_5 T^3 + 12a_4 T^2 + 20a_5 T^3 = 0
\]

\[
40a_5 T^3 = -4a_4 T^2 \quad \Rightarrow \quad a_5 = -\frac{2}{5} \frac{1}{T^3} a_4
\]

\[
(x_0 - x_T) = -2a_3 T^3 - 2a_4 T^4 - 2a_5 T^5
\]

\[
* = \frac{a_4}{3} 2a_4 T^3 - \frac{5}{2} \left(-\frac{2}{5} \frac{1}{T^3} a_4\right) T^4
\]

\[
3a_3 T^2 = -\frac{4}{3} a_4 T^2 + \frac{2}{3} T^3 a_4
\]
\[ 23 T^2 = -\frac{2}{3} 24 T^3 \]

\[
(x - x_T) = +\frac{2}{3} 24 T^4 - 24 T^4 + 2 \frac{1}{5} \frac{24 T^6}{T^5} = \frac{10 - 15 + 6}{15} 24 T^4
\]

\[ 24 = 15 \frac{(x_0 - x_T)}{T^5} \]

\[ 23 = -\frac{2}{3} T \cdot \frac{15}{T^4} (x - x_T) = -10 \frac{(x_0 - x_T)}{T^3} \]

\[ 25 = -\frac{2}{5} \frac{1}{T} \frac{15 (x_0 - x_T)}{T^4} = -6 \frac{(x_0 - x_T)}{T^4} \]

\[ x(t) = x_0 - \frac{10 (x_0 - x_T)}{T^2} t^2 + 75 \frac{(x_0 - x_T)}{T^4} t^4 - \frac{6 (x_0 - x_T)}{T^5} t^5 \]

MINIMUM JUNK TASTE TODAY