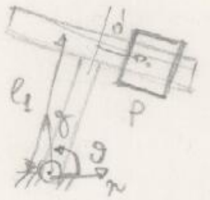
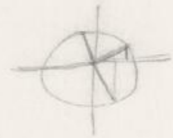
22/01/2010



$$q = \begin{bmatrix} \vartheta \\ d \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad , \quad \tau = \begin{bmatrix} \tau_\vartheta \\ F_d \end{bmatrix}$$

$L = K - V$, $V = 0$ poichè ci troviamo sul piano orizzontale

$$K = K_1 + K_2$$



$$K_2 = \frac{1}{2} m_2 v_2^T v_2 \qquad v_2 = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = J_p \cdot \dot{q}$$

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} l_1 c_\vartheta + d\, c_{\vartheta - \frac{\pi}{2}} \\ l_1 s_\vartheta + d\, s_{\vartheta - \frac{\pi}{2}} \end{bmatrix} = \begin{bmatrix} l_1 c_\vartheta + d s_\vartheta \\ l_1 s_\vartheta - d c_\vartheta \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -l_1 s_\vartheta + d c_\vartheta & + s_\vartheta \\ l_1 c_\vartheta + d s_\vartheta & - c_\vartheta \end{bmatrix} \dot{q} =$$

$$K_2 = \frac{1}{2} m_2 \dot{q}^T \begin{bmatrix} -l_1 s_\vartheta + d c_\vartheta & l_1 c_\vartheta + d s_\vartheta \\ s_\vartheta & -c_\vartheta \end{bmatrix} \begin{bmatrix} -l_1 s_\vartheta + d c_\vartheta & s_\vartheta \\ l_1 c_\vartheta + d s_\vartheta & -c_\vartheta \end{bmatrix} \dot{q}$$

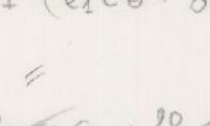
$$= \frac{1}{2} m_2 \dot{q}^T \begin{bmatrix} (d c_\vartheta - l_1 s_\vartheta)^2 + (l_1 c_\vartheta + d s_\vartheta)^2 & s_\vartheta(-l_1 s_\vartheta + d c_\vartheta) - c_\vartheta(l_1 c_\vartheta + d s_\vartheta) \\ = & s_\vartheta^2 + c_\vartheta^2 \end{bmatrix} \dot{q}$$

$$= \frac{1}{2} m_2 \dot{q}^T \begin{bmatrix} d^2 + l_1^2 - \cancel{2 d l_1 c_\vartheta s_\vartheta} + \cancel{2 l_1 d c_\vartheta s_\vartheta} & -l_1 \\ -l_1 & 1 \end{bmatrix} \dot{q}$$

$$K_1 = \frac{1}{2} I_1 \dot{q}_1^2 = \frac{1}{2} \dot{q}^T \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \dot{q} \qquad I_1 = I_A + I_B$$

$$I_A = I_{A,CM} + m_A \left(\frac{l_1}{2}\right)^2 \qquad I_B = I_{B,CM} + m_B l_1^2$$

$$I_{A,CM} = m_A \frac{r_A^2}{4} + m_A \frac{l_A^2}{12}$$



$$K = \frac{1}{2} \dot{q}^T \begin{bmatrix} I_1 + m_2 (d^2 + l_1^2) & -m_2 l_1 \\ -m_2 l_1 & m_2 \end{bmatrix} \dot{q}$$

$$= \frac{1}{2} \left(I_1 + m_2 d^2 + m_2 l_1^2\right) \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 - m_2 l_1 \dot{q}_1 \dot{q}_2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} 0 \\ m_2 d \dot{q}_1^2 \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} (I_1 + m_2 d^2 + m_2 l_1^2)\dot{q}_1 - m_2 l_1 \dot{q}_2 \\ m_2 \dot{q}_2 - m_2 \dot{q}_1 l_1 \end{bmatrix}$$

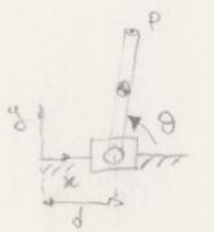
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} (I_1 + m_2 d^2 + m_2 l_1^2)\ddot{q}_1 - m_2 l_1 \ddot{q}_2 \\ m_2 \ddot{q}_2 - m_2 \ddot{q}_1 l_1 \end{bmatrix} + \begin{bmatrix} 2 m_2 d \dot{d} \dot{q}_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (I_1 + m_2 d^2 + m_2 l_1^2) & -m_2 l_1 \\ -m_2 l_1 & m_2 \end{bmatrix} \ddot{q} + \begin{bmatrix} 2 m_2 \dot{d} d \dot{q}_1 \\ 0 \end{bmatrix}$$

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} 2 m_2 \dot{d} \dot{q}_1 d \\ - m_2 d \dot{q}_1^2 \end{bmatrix} = \begin{bmatrix} m_2 \dot{d} d & m_2 \dot{q}_1 d \\ -m_2 d \dot{q}_1 & 0 \end{bmatrix} \dot{q}$$

$$\dot{M} - 2C = \begin{bmatrix} 2 m_2 d \dot{d} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 m_2 d \dot{d} & 2 m_2 \dot{q}_1 d \\ -2 m_2 d \dot{q}_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 m_2 \dot{q}_1 d \\ -2 m_2 d \dot{q}_1 & 0 \end{bmatrix}$$



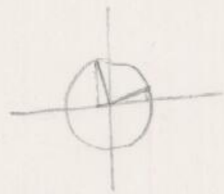
$$K = K_1 + K_2$$

$$K_1 = \frac{1}{2} m_1 \cdot v_1^T v_1$$

$$K_2 = \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 v_{CM,2}^T v_{CM}$$

$$\to I_2 = I_{2,CM}$$

$$q = \begin{bmatrix} d \\ \theta \end{bmatrix}, \quad \tau = \begin{bmatrix} F \\ \tau_\theta \end{bmatrix}$$



$$v_1 = \dot{d}, \quad K_1 = \frac{1}{2} m_1 \dot{d}^2$$

$$v_{CM,2}, \quad P_{CM,2} = \begin{bmatrix} d \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{l_2}{2} c\theta \\ \frac{l_2}{2} s\theta \end{bmatrix} = \begin{bmatrix} d + \frac{l_2}{2} c\theta \\ \frac{l_2}{2} s\theta \end{bmatrix}$$

$$v_{CM,2} = \begin{bmatrix} 1 & -\frac{l_2}{2} s\theta \\ 0 & \frac{l_2}{2} c\theta \end{bmatrix} \dot{q}$$

$$K_2 = \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{q}^T \begin{bmatrix} 1 & 0 \\ -\frac{l_2}{2} s\theta & \frac{l_2}{2} c\theta \end{bmatrix} \begin{bmatrix} 1 & -\frac{l_2}{2} s\theta \\ 0 & \frac{l_2}{2} c\theta \end{bmatrix} \dot{q}$$

$$= \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{q}^T \begin{bmatrix} 1 & -\frac{l_2}{2} s\theta \\ -\frac{l_2}{2} s\theta & \frac{l_2^2}{4} \end{bmatrix} \dot{q}$$

$$V = m_2 \cdot g \cdot h = m_2 g \cdot \frac{l_2}{2} s\theta$$

$$L = K - V = \frac{1}{2} m_1 \dot{d}^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \dot{d}^2 + \frac{1}{2} m_2 \frac{l_2^2}{4} \dot{\theta}_2^2 - \frac{1}{2} m_2 l_2 s\theta \dot{d}\dot{\theta} - m_2 g \frac{l_2}{2} s\theta$$

$$= \frac{1}{2} \dot{q}^T \begin{bmatrix} m_1 + m_2 & -\frac{1}{2} m_2 l_2 s\theta \\ -\frac{1}{2} m_2 l_2 s\theta & I_2 + \frac{1}{2} m_2 \frac{l_2^2}{4} \end{bmatrix} \dot{q} - m_2 g \frac{l_2}{2} s\theta$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} 0 \\ -\frac{1}{2} m l_2 c\theta \dot{d}\dot{\theta} - m_2 g \frac{l_2}{2} c\theta \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} m_1 \dot{d} + \cancel{I_2 \dot{\theta}_2} + m_2 \dot{d} + \cancel{m_2 \frac{l_2^2}{4} \dot{\theta}_2} - \frac{1}{2} m_2 l_2 s\theta \dot{\theta} \\ \left(I_2 + m_2 \frac{l_2^2}{4}\right) \dot{\theta}_2 - \frac{1}{2} m_2 l_2 s\theta \dot{d} \end{bmatrix}$$

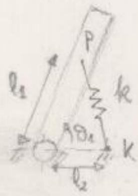
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \begin{bmatrix} (m_1 + m_2) \ddot{d} - m_2 l_2 s\theta \ddot{\theta} - \frac{1}{2} m_2 l_2 c\theta \dot{\theta}^2 \\ \left(I_2 + m_2 \frac{l_2^2}{4}\right) \ddot{\theta}_2 - \frac{1}{2} m_2 l_2 s\theta \ddot{d} - m_2 l_2 c\theta \dot{\theta} \dot{d} \end{bmatrix}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \begin{bmatrix} m_1 + m_2 & -\frac{1}{2} m_2 l_2 s\theta \\ -\frac{1}{2} m_2 l_2 s\theta & I_2 + m_2 \frac{l_2^2}{4} \end{bmatrix} \ddot{q} + \begin{bmatrix} -m_2 l_2 c\theta \dot{\theta}^2 \\ -m_2 l_2 c\theta \dot{\theta} \dot{d} \end{bmatrix} + \begin{bmatrix} 0 \\ + m_2 l_2 c\theta \dot{d}\dot{\theta} + m_2 g \frac{l_2}{2} c\theta \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -m_2 l_2 c\theta \dot\theta \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ m_2 g \frac{l_2}{2} c\theta \end{bmatrix}$$

$$\dot M - 2C = \begin{bmatrix} 0 & -m_2 l_2 c\theta \dot\theta \\ -m_2 l_2 c\theta \dot\theta & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & -m_2 l_2 c\theta \dot\theta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m_2 l_2 c\theta \dot\theta \\ -m_2 l_2 c\theta \dot\theta & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 + m_2 & -\frac{1}{2} m_2 l_2 s\theta \\ -\frac{1}{2} m_2 l_2 s\theta & I_2 + m_2 \frac{l_2^2}{4} \end{bmatrix}$$



$$K = \frac{1}{2} I_1 \dot\theta_1^2$$

$$V = \frac{1}{2} k\, l_k^2$$

$$P = \begin{bmatrix} l_1 c\theta \\ l_1 s\theta \end{bmatrix}, \quad K = \begin{bmatrix} l_2 \\ 0 \end{bmatrix} \qquad PK = \begin{bmatrix} -l_2 \\ 0 \end{bmatrix} + \begin{bmatrix} l_1 c\theta \\ l_1 s\theta \end{bmatrix} = \begin{bmatrix} l_1 c\theta - l_2 \\ l_1 s\theta \end{bmatrix}$$

$$\|PK\|^2 = l_k^2 = l_2^2 + l_1^2 - 2 l_1 l_2 c\theta$$

$$L = K - V = \frac{1}{2} I_1 \dot\theta_1^2 - \frac{1}{2} k \left(l_2^2 + l_1^2 - 2 l_1 l_2 c\theta \right)$$